THE ART OF COMPUTER PROGRAMMING

PRE-FASCICLE 4B

A DRAFT OF SECTION 7.2.1.7: HISTORY OF COMBINATORIAL GENERATION

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See also http://www-cs-faculty.stanford.edu/~knuth/sgb.html for information about *The Stanford GraphBase*, including downloadable software for dealing with the graphs used in many of the examples in Chapter 7.

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PREFACE

I like to work in a variety of fields
in order to spread my mistakes more thinly.
— VICTOR KLEE (1999)

THIS BOOKLET contains draft material that I'm circulating to experts in the field, in hopes that they can help remove its most egregious errors before too many other people see it. I am also, however, posting it on the Internet for courageous and/or random readers who don't mind the risk of reading a few pages that have not yet reached a very mature state. *Beware:* This material has not yet been proofread as thoroughly as the manuscripts of Volumes 1, 2, and 3 were at the time of their first printings. And those carefully-checked volumes, alas, were subsequently found to contain thousands of mistakes.

Given this caveat, I hope that my errors this time will not be so numerous and/or obtrusive that you will be discouraged from reading the material carefully. I did try to make it both interesting and authoritative, as far as it goes. But the field is so vast, I cannot hope to have surrounded it enough to corral it completely. Therefore I beg you to let me know about any deficiencies you discover.

To put the material in context, this is Section 7.2.1.7 of a long, long chapter on combinatorial algorithms. Chapter 7 will eventually fill three volumes (namely Volumes 4A, 4B, and 4C), assuming that I'm able to remain healthy. It will begin with a short review of graph theory, with emphasis on some highlights of significant graphs in The Stanford GraphBase, from which I will be drawing many examples. Then comes Section 7.1, which deals with the topic of bitwise manipulations. (I drafted about 60 pages about that subject in 1977, but those pages need extensive revision; meanwhile I've decided to work for awhile on the material that follows it, so that I can get a better feel for how much to cut.) Section 7.2 is about generating all possibilities, and it begins with Section 7.2.1: Generating Basic Combinatorial Patterns — which, in turn, begins with Section 7.2.1.1, "Generating all n-tuples," Section 7.2.1.2, "Generating all permutations," . . . , Section 7.2.1.6, "Generating all trees." (Readers of the present booklet should have already looked at those sections, drafts of which are available as Pre-Fascicles 2A, 2B, 3A, 3B, and 4A.) The stage is now set for the main contents of this booklet, Section 7.2.1.7: "History and further references." Section 7.2.2 will deal with backtracking in general. And so it will continue, if all goes well; an outline of the entire Chapter 7 as currently envisaged appears on the `taocp` webpage that is cited on page ii.

Writing about history is extraordinarily difficult, not only because the source materials are widely scattered but also because I must operate at the limit of my ability to understand languages other than English. Furthermore, facts about real life are much more complicated than facts about mathematics. No summary can adequately convey the true feelings of an era or the true spirit of a culture, yet the story that I'm trying to tell in this section covers many centuries of development in many different parts of the world. The story is fascinating, and many parts of it do not seem to have been told before, at least not in English. Therefore I'm keen to have professional historians of mathematics take a look at what I've been able to piece together, hoping that they will not be too shocked by blunders that have resulted from my present ignorance and/or incompetence. I hope also to get advice from people of many different cultures who know of relevant traditions that have not yet been well studied by professional historians.

The answer to exercise 6 poses two historical problems that I haven't been able to resolve. I urgently need your help also with respect to some exercises that I made up as I was preparing this material. I certainly don't like to get credit for things that have already been published by others, and most of these results are quite natural "fruits" that were just waiting to be "plucked." Therefore please tell me if you know who I should have credited, with respect to the ideas found in exercises 2, 8, 10, 16, 19, 15, and/or 26.

I shall happily pay a finder's fee of $2.56 for each error in this draft when it is first reported to me, whether that error be typographical, technical, or historical. The same reward holds for items that I forgot to put in the index. And valuable suggestions for improvements to the text are worth 32¢ each. (Furthermore, if you find a better solution to an exercise, I'll actually reward you with immortal glory instead of mere money, by publishing your name in the eventual book:–)

Cross references to yet-unwritten material sometimes appear as '00'; this impossible value is a placeholder for the actual numbers to be supplied later.

Happy reading!

Stanford, California D. E. K.
12 October 2004

[This subject] has a relation
to almost every species of useful knowledge
that the mind of man can be employed upon.
— JAMES BERNOULLI, *Ars Conjectandi* (1713)

7.2.1.7. History and further references. Early work on the generation of combinatorial patterns began as civilization itself was taking shape. The story is quite fascinating, and we will see that it spans many cultures in many parts of the world, with ties to poetry, music, and religion. There is space here to discuss only some of the principal highlights; but perhaps a few glimpses into the past will stimulate the reader to dig deeper into the roots of the subject, as the world gets ever smaller and as global scholarship continues to advance.

Lists of binary n-tuples can be traced back thousands of years to ancient China, India, and Greece. The most notable source — because it still is a best-selling book in modern translations — is the Chinese *I Ching* or *Yijing*, whose name means "the Bible of Changes." This book, which is one of the five classics of Confucian wisdom, consists essentially of $2^6 = 64$ chapters; and each chapter is symbolized by a hexagram formed from six lines, each of which is either ⚋ ("yin") or ⚊ ("yang"). For example, hexagram 1 is pure yang, ䷀; hexagram 2 is pure yin, ䷁; and hexagram 64 intermixes yin and yang, with yang on top: ䷿. Here is the complete list:

1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
䷀	䷁	䷂	䷃	䷄	䷅	䷆	䷇	䷈	䷉	䷊	䷋	䷌	䷍	䷎	䷏
17	18	19	20	21	22	23	24	25	26	27	28	29	30	31	32
䷐	䷑	䷒	䷓	䷔	䷕	䷖	䷗	䷘	䷙	䷚	䷛	䷜	䷝	䷞	䷟
33	34	35	36	37	38	39	40	41	42	43	44	45	46	47	48
䷠	䷡	䷢	䷣	䷤	䷥	䷦	䷧	䷨	䷩	䷪	䷫	䷬	䷭	䷮	䷯
49	50	51	52	53	54	55	56	57	58	59	60	61	62	63	64
䷰	䷱	䷲	䷳	䷴	䷵	䷶	䷷	䷸	䷹	䷺	䷻	䷼	䷽	䷾	䷿

(1)

This arrangement of the 64 possibilities is called King Wen's ordering, because the basic text of the *I Ching* has traditionally been ascribed to King Wen (c. 1100 B.C.), the legendary progenitor of the Chou dynasty. Ancient texts are, however, notoriously difficult to date reliably, and modern historians have found no solid evidence that anyone actually compiled such a list of hexagrams before the third century B.C.

Notice that the hexagrams of (1) occur in pairs: Those with odd numbers are immediately followed by their top-to-bottom reflections, except when reflection would make no change; and the eight symmetrical diagrams are paired with their complements ($1 = \overline{2}$, $27 = \overline{28}$, $29 = \overline{30}$, $61 = \overline{62}$). Hexagrams that are composed from two trigrams that represent the four basic elements heaven (☰), earth (☷), fire (☲), and water (☵) have also been placed judiciously. Otherwise the arrangement appears to be essentially random, as if a person untrained in mathematics kept listing different possibilities until being unable to come up with any more. A few intriguing patterns do exist between the pairs, but no more than are present by coincidence in the digits of π (see 3.3–(1)).

Yin and yang represent complementary aspects of the elementary forces of nature, always in tension, always changing. The *I Ching* is somewhat analogous to a thesaurus in which the hexagrams serve as an index to accumulated wisdom about fundamental concepts like giving (䷀), receiving (䷁), modesty (䷎), joy (䷹), fellowship (䷌), withdrawal (䷠), peace (䷊), conflict (䷅), organization (䷆), corruption (䷑), immaturity (䷃), elegance (䷕), etc. One can choose a pair of hexagrams at random, obtaining the second from the first by, say, independently changing each yin to yang (or vice versa) with probability 1/4; this technique yields 4096 ways to ponder existential mysteries, as well as a Markov process by which change itself might perhaps give meaning to life.

A strictly logical way to arrange the hexagrams was eventually introduced about A.D. 1060 by Shao Yung. His ordering, which proceeded lexicographically from ䷁ to ䷖ to ䷇ to ䷓ to ䷏ to $\cdots$ to ䷪ to ䷀ (reading each hexagram from bottom to top), was much more user-friendly than the King Wen order, because a random pattern could now be found quickly. When G. W. Leibniz learned about this sequence of hexagrams in 1702, he jumped to the erroneous conclusion that Chinese mathematicians had once been familiar with binary arithmetic. [See Frank Swetz, *Mathematics Magazine* **76** (2003), 276–291. Further details about the *I Ching* can be found, for example, in Joseph Needham's *Science and Civilisation in China* **2** (Cambridge University Press, 1956), 304–345; R. J. Lynn, *The Classic of Changes* (New York: Columbia University Press, 1994).]

Another ancient Chinese philosopher, Yang Hsiung, proposed a system based on 81 ternary tetragrams instead of 64 binary hexagrams. His *Canon of Supreme Mystery*, written c. 2 B.C., has recently been translated into English by Michael Nylan (Albany, New York: 1993). Yang described a complete, hierarchical ternary tree structure in which there are 3 regions, with 3 provinces in each region, 3 departments in each province, 3 families in each department, and 9 short poems called "appraisals" for each family, hence 729 appraisals in all — making almost exactly 2 appraisals for every day in the year. His tetragrams were arranged in strict lexicographic order when read top-to-bottom: 𝌆, 𝌇, 𝌈, 𝌉, 𝌊, 𝌋, 𝌌, . . . , 𝍖. In fact, as explained on page 28 of Nylan's book, Yang presented a simple way to compute the rank of each tetragram, as if using a radix-3 number system. Thus he would not have been surprised or impressed by Shao Yung's systematic ordering of binary hexagrams, although Shao lived more than 1000 years later.

Indian prosody. Binary n-tuples were studied in a completely different context by pundits in ancient India, who investigated the poetic meters of sacred Vedic chants. Syllables in Sanskrit are either short (।) or long (ऽ), and the study of syllable patterns is called "prosody." Modern writers use the symbols $\smile$ and — instead of । and ऽ. A typical Vedic verse consists of four lines with n syllables per line, for some $n \geq 8$; prosodists therefore sought a way to classify all 2^n possibilities. The classic work *Chandaḥśāstra* by Piṅgala, written before A.D. 400 and probably much earlier (the exact date is quite uncertain), described procedures by which one could readily find the index k of any given pattern of $\smile$s and —s, as well as to find the kth pattern, given k. In other words, Piṅgala explained how to *rank* any given pattern as well as to *unrank* any given index;

thus he went beyond the work of Yang Hsiung, who had considered ranking but not unranking. Piṅgala's methods were also related to exponentiation, as we have noted earlier in connection with Algorithm 4.6.3A.

The next important step was taken by a prosodist named Kedāra in his work *Vṛttaratnākara*, thought to have been written in the 8th century. Kedāra gave a step-by-step procedure for listing all the n-tuples from ———...— to ⌣——...— to —⌣—...— to ⌣⌣—...— to ——⌣...— to ⌣—⌣...— to $\cdots$ to ⌣⌣⌣...⌣, essentially Algorithm 7.2.1.1M in the case of radix 2. His method may well have been the first-ever explicit algorithm for combinatorial sequence generation. [See B. van Nooten, *J. Indian Philos.* **21** (1993), 31–50.]

Poetic meters can also be regarded as rhythms, with one beat for each ⌣ and two beats for each —. An n-syllable pattern can involve between n and $2n$ beats, but musical rhythms suitable for marching or dancing generally are based on a fixed number of beats. Therefore it was natural to consider the set of all sequences of ⌣s and —s that have exactly m beats, for fixed m. Such patterns are now called Morse code sequences of length m, and we know from exercise 4.5.3–32 that there are exactly F_{m+1} of them. For example, the 21 sequences when $m = 7$ are

⌣———, —⌣——, ⌣⌣⌣——, ——⌣—, ⌣⌣—⌣—,
⌣—⌣⌣—, —⌣⌣⌣—, ⌣⌣⌣⌣⌣—, ———⌣,
⌣⌣——⌣, ⌣—⌣—⌣, —⌣⌣—⌣, ⌣⌣⌣⌣—⌣, (2)
⌣——⌣⌣, —⌣—⌣⌣, ⌣⌣⌣—⌣⌣, ——⌣⌣⌣,
⌣⌣—⌣⌣⌣, ⌣—⌣⌣⌣⌣, —⌣⌣⌣⌣⌣, ⌣⌣⌣⌣⌣⌣⌣.

In this way Indian prosodists were led to discover the Fibonacci sequence, as we have observed in Section 1.2.8.

Moreover, the anonymous author of *Prākṛta Paiṅgala* (c. 1320) discovered elegant algorithms for ranking and unranking with respect to m-beat rhythms. To find the kth pattern, one starts by writing down m ⌣s, then expresses the difference $d = F_{m+1} - k$ as a sum of Fibonacci numbers $F_{j_1} + \cdots + F_{j_t}$; here F_{j_1} is the largest Fibonacci number that is $\le d$ and F_{j_2} is the largest $\le d - F_{j_1}$, etc., continuing until the remainder is zero. Then beats $j-1$ and j are to be changed from ⌣⌣ to —, for $j = j_1, \ldots, j_t$. For example, to get the 5th element of (2) we compute $21 - 5 = 16 = 13 + 3 = F_7 + F_4$; the answer is ⌣⌣—⌣—.

A few years later, Nārāyaṇa Paṇḍita treated the more general problem of finding all compositions of m whose parts are $\le q$, where q is *any* given positive integer. As a consequence he discovered the qth-order Fibonacci sequence 5.4.2–(4), which was destined to be used 600 years later in polyphase sorting; he also developed the corresponding ranking and unranking algorithms. [See Parmanand Singh, *Historia Mathematica* **12** (1985), 229–244, and exercise 15.]

Piṅgala gave special code-names to all the three-syllable meters,

——— = म (m),	——⌣ = त (t),	
⌣—— = य (y),	⌣—⌣ = ज (j),	(3)
—⌣— = र (r),	—⌣⌣ = भ (bh),	
⌣⌣— = स (s),	⌣⌣⌣ = न (n),	

and students of Sanskrit have been expected to memorize them ever since. Somebody long ago devised a clever way to recall these codes, by inventing the nonsense word *yamātārājabhānasalagām* (यमाताराजभानसलगाम); the point is that the ten syllables of this word can be written

ya	mā	tā	rā	ja	bhā	na	sa	la	gām	
◡	—	—	—	◡	—	◡	◡	◡	—	(4)

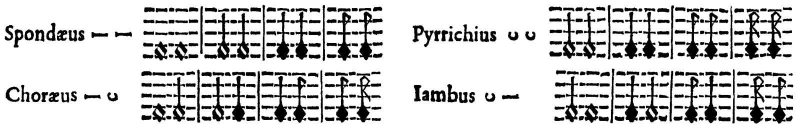
and each three-syllable pattern occurs just after its code name. The origin of yamā . . . lagām is obscure, but Subhash Kak [*Indian J. History of Science* **35** (2000), 123–127] has traced it back at least to C. P. Brown's *Sanskrit Prosody* (1869), page 28; thus it qualifies as the earliest known appearance of a "de Bruijn cycle" that encodes binary n-tuples.

Meanwhile, in Europe. In a similar way, classic Greek poetry was based on groups of short and/or long syllables called "metrical feet," analogous to bars of music. Each basic type of foot acquired a Greek name; for example, two short syllables '◡◡' were called a *pyrrhic*, and two long syllables '——' were called a *spondee*, because those rhythms were used respectively in a song of war (πυρρίχη) or a song of peace (σπονδαί). Greek names for metric feet were soon assimilated into Latin and eventually into modern languages, including English:

◡	arsis	◡◡◡◡	proceleusmatic	
—	thesis	◡◡◡—	fourth pæon	
		◡◡—◡	third pæon	
◡◡	pyrrhic	◡◡——	minor ionic	
◡—	iambus	◡—◡◡	second pæon	
—◡	trochee	◡—◡—	diiambus	
——	spondee	◡——◡	antispast	
		◡———	first epitrite	
◡◡◡	tribrach	—◡◡◡	first pæon	(5)
◡◡—	anapest	—◡◡—	choriambus	
◡—◡	amphibrach	—◡—◡	ditrochee	
◡——	bacchius	—◡——	second epitrite	
—◡◡	dactyl	——◡◡	major ionic	
—◡—	amphimacer	——◡—	third epitrite	
——◡	palimbacchius	———◡	fourth epitrite	
———	molossus	————	dispondee	

Alternative names, like "choree" instead of "trochee," or "cretic" instead of "amphimacer," were also in common use. Moreover, by the time Diomedes wrote his Latin grammar (approximately A.D. 375), each of the 32 *five*-syllable feet had acquired at least one name. Diomedes also pointed out the relation between complementary patterns; he stated for example that tribrach and molossus are "*contrarius*," as are amphibrach and amphimacer. But he also regarded dactyl as the contrary of anapest, and bacchius as the contrary of palimbacchius, although the literal meaning of *palimbacchius* is actually "reverse bacchius." Greek prosodists had no standard order in which to list the individual possibilities, and the form of the names makes it clear that no connection to a radix-two number system was contemplated. [See H. Keil, *Grammatici Latini* **1** (1857), 474–482; W. von Christ, *Metrik der Griechen und Römer* (1879), 78–79.]

Surviving fragments of a work by Aristoxenus called *Elements of Rhythm* (c. 325 B.C.) show that the same terminology was applied also to music. And indeed, the same traditions lived on after the Renaissance; for example, we find



on page 32 of Athanasius Kircher's *Musurgia Universalis* **2** (Rome: 1650), and Kircher went on to describe all of the three-note and four-note rhythms of (5).

Early lists of permutations. We've traced the history of formulas for counting permutations in Section 5.1.2; but nontrivial *lists* of permutations were not published until hundreds of years after the formula $n!$ was discovered. The first such tabulation currently known was compiled by the Italian physician Shabbetai Donnolo in his commentary on the kabbalistic *Sefer Yetzirah*, written in A.D. 946. Table 1 shows his list for $n = 5$ as it was subsequently printed in Warsaw (1884). (The Hebrew letters in this table are typeset in a rabbinical font traditionally used for commentaries; notice that the letter מ changes its shape to ם when it appears at the left end of a word.) Donnolo went on to list 120 permutations of the six-letter word שכתבנו, all beginning with shin (ש); then he noted that 120 more could be obtained with each of the other five letters in front, making 720 in all. His lists involved groupings of six permutations, but in a haphazard fashion that led him into error (see exercise 4). Although he knew how many permutations there were supposed to be, and how many should start with a given letter, he evidently didn't have an algorithm for generating them.

Table 1

A MEDIEVAL LIST OF PERMUTATIONS

דברים, דבירם, דבימר, דבמיר, דבמרי, דברמי, דרבים, דרבמי, דרימב, דריבם, דרמבי, דרמיב,
דיברם, דירבם, דירמב, דיבמר, דימרב, דימבר, דמביר, דמברי, דמרבי, דמריב, דמיבר, דמירב,
בדימר, בדירם, בדרמי, בדרים, בדמרי, בדמיר, בירמד, בידרם, בירדם, בימדר, בימרד, בידמר,
במדרי, במדיר, במריד, במרדי, במדיר, במירד, בדמיר, בדמרי, בדרמי, בדרים, בדימר, בדירם,
רימדב, רימבד, רידבם, רידמב, ריבדם, ריבמד, רדמבי, רדבמי, רדיבם, רדימב, רדבים, רדמיב,
רבמדי, רבדמי, רבדים, רבידם, רבימד, רבמיד, רמידב, רמיבד, רמדיב, רמביד, רמבדי, רמדבי,
ימדרב, ימרדב, ימבדר, ימברד, ימרבד, ימדבר, ידברם, ידרבם, ידבמר, ידמבר, ידרמב, ידמרב,
יבדרם, יבדמר, יברדם, יברמד, יבמדר, יבמרד, ירדבם, ירדמב, ירמבד, ירבמד, ירבדם, ירדמב,
מדברי, מדרבי, מדביר, מדיבר, מדירב, מדריב, מבדיר, מבידר, מבריד, מבירד, מברדי, מבדרי,
מרדבי, מרבדי, מרדיב, מרביד, מריבד, מרידב, מידבר, מידרב, מיברד, מיבדר, מירדב, מירבד

A complete list of all 720 permutations of $\{a, b, c, d, e, f\}$ appeared on pages 668–671 of Jeremias Drexel's *Orbis Phaëthon* (Munich: 1629; also on pages 526–531 of the Cologne edition in 1631). He offered it as proof that a man with six guests could seat them differently at lunch and dinner every day for a year — altogether 360 days, because there were five days of fasting during Holy Week. Shortly afterwards, Marin Mersenne exhibited all 720 permutations of the six tones {ut, re, mi, fa, sol, la}, on pages 111–115 of his *Traitez de la Voix et des Chants* (Volume 2 of *Harmonie Universelle*, 1636); then on pages 117–128 he presented the same data in musical notation:

Drexel's table was organized lexicographically by columns; Mersenne's tables were lexicographic with respect to the order ut < re < mi < fa < sol < la, beginning with "ut,re,mi,fa,sol,la" and ending with "la,sol,fa,mi,re,ut." Mersenne also prepared a 674-page manuscript that listed all 40,320 permutations of *eight* notes [Bibliothèque nationale de France, Fonds Français, no. 24256].

We saw in Section 7.2.1.2 that the important idea of plain changes, Algorithm 7.2.1.2P, was invented in England a few years later.

Methods for listing all permutations of a multiset with *repeated* elements were often misunderstood by early authors. For example, when Bhāskara exhibited the permutations of $\{4,5,5,5,8\}$ in section 271 of his *Līlāvatī* (c. 1150), he gave them in the following order:

४८५५५	८४५५५	५४८५५	५८४५५	५५४८५	
५५८४५	५५५४८	५५५८४	४५८५५	४५५८५	(6)
४५५५८	८५४५५	८५५४५	८५५५४	५४५८५	
५८५४५	५५४५८	५५८५४	५४५५८	५८५५४	

Mersenne used a slightly more sensible but not completely systematic order on page 131 of his book when he listed sixty anagrams of the Latin name IESVS. When Athanasius Kircher wanted to illustrate the 30 permutations of a five-note melody on pages 10 and 11 of *Musurgia Universalis* **2** (1650), this lack of a system got him into trouble (see exercise 5):


(7)

But John Wallis knew better. On page 117 of his *Discourse of Combinations* (1685) he correctly listed the 60 anagrams of "messes" in lexicographic order, if we let m < e < s; and on page 126 he recommended respecting alphabetic order "that we may be the more sure, not to miss any."

We will see later that the Indian mathematician Nārāyaṇa Paṇḍita had already developed a theory of permutation generation in the 14th century, although his work remained almost totally unknown.

Seki's list. Takakazu Seki (1642–1708) was a charismatic teacher and researcher who revolutionized the study of mathematics in 17th-century Japan. While he was studying the elimination of variables from simultaneous homogeneous equations, he was led to expressions such as $a_1b_2 - a_2b_1$ and $a_1b_2c_3 - a_1b_3c_2 + a_2b_3c_1 - a_2b_1c_3 + a_3b_1c_2 - a_3b_2c_1$, which we now recognize as *determinants.* In 1683 he published a booklet about this discovery, introducing an ingenious scheme for listing all permutations in such a way that half of them were "alive" (even) and the other half were "dead" (odd). Starting with the case $n = 2$, when '12' was alive and '21' was dead, he formulated the following rules for $n > 2$:

1) Take every live permutation for $n-1$, increase all its elements by 1, and insert 1 in front. This rule produces $(n-1)!/2$ "basic permutations" of $\{1, \ldots, n\}$.
2) From each basic permutation, form $2n$ others by rotation and reflection:

$$a_1a_2\ldots a_{n-1}a_n,\ a_2\ldots a_{n-1}a_na_1,\ \ldots,\ a_na_1a_2\ldots a_{n-1}; \tag{8}$$
$$a_na_{n-1}\ldots a_2a_1,\ a_1a_na_{n-1}\ldots a_2,\ \ldots,\ a_{n-1}\ldots a_2a_1a_n. \tag{9}$$

If n is odd, those in the first row are alive and those in the second row are dead; if n is even, those in each row are alternatively alive, dead, ..., dead.

For example, when $n = 3$ the only basic permutation is 123. Thus 123, 231, 312 are alive while 321, 132, 213 are dead, and we've successfully generated the six terms of a 3×3 determinant. The basic permutations for $n = 4$ are 1234, 1342, 1423; and from, say, 1342 we get a set of eight, namely

$$+1342 - 3421 + 4213 - 2134 + 2431 - 1243 + 3124 - 4321, \tag{10}$$

alternately alive (+) and dead (−). A 4×4 determinant therefore includes the terms $a_1b_3c_4d_2 - a_3b_4c_2d_1 + \cdots - a_4b_3c_2d_1$ and sixteen others.

Seki's rule for permutation generation is quite pretty, but unfortunately it has a serious problem: It doesn't work when $n > 4$. His error seems to have gone unrecognized for hundreds of years. [See Y. Mikami, *The Development of Mathematics in China and Japan* (1913), 191–199; *Takakazu Seki's Collected Works* (Osaka: 1974), 18–20, 八五一 .. 一四一; and exercises 7–8.]

Lists of combinations. The earliest exhaustive list of *combinations* known to have survived the ravages of time appears in the last book of Suśruta's well-known Sanskrit treatise on medicine, Chapter 63, written before A.D. 600 and perhaps much earlier. Noting that medicine can be sweet, sour, salty, peppery, bitter, and/or astringent, Suśruta's book diligently listed the $(15, 20, 15, 6, 1, 6)$ cases that arise when those qualities occur two, three, four, five, six, and one at a time.

Bhāskara repeated this example in sections 110–114 of *Līlāvatī*, and observed that the same reasoning applies to six-syllable poetic meters with a given number of long syllables. But he simply mentioned the totals, $(6, 15, 20, 15, 6, 1)$, without listing the combinations themselves. In sections 274 and 275, he observed that the numbers $(n)(n-1)\ldots(n-k+1)/(k(k-1)\ldots(1))$ enumerate *compositions* (that is, ordered partitions) as well as combinations; again he gave no list.

To avoid prolixity this is treated in a brief manner;
for the science of calculation is an ocean without bounds.
— *Bhāskara* (c. 1150)

An isolated but interesting list of combinations appeared in the remarkable algebra text *Al-Bāhir fi'l-ḥisāb* (*The Shining Book of Calculation*), written by al-Samaw'al of Baghdad when he was only 19 years old (1144). In the closing part of that work he presented a list of $\binom{10}{6} = 210$ simultaneous linear equations in 10 unknowns:

Al-Samaw'al's Arabic original

٦٥	٦٥٤٣٢١	ا
٧٠	٧٥٤٣٢١	ب
٧٥	٨٥٤٣٢١	ج
	⋮	
٩١	١٠٩٨٧٦٤	زط
١٠٠	١٠٩٨٧٦٥	زي

Equivalent modern notation

$$\begin{aligned} &(1)\quad x_1 + x_2 + x_3 + x_4 + x_5 + x_6 = 65\\ &(2)\quad x_1 + x_2 + x_3 + x_4 + x_5 + x_7 = 70\\ &(3)\quad x_1 + x_2 + x_3 + x_4 + x_5 + x_8 = 75\\ &\qquad\vdots\\ &(209)\; x_4 + x_6 + x_7 + x_8 + x_9 + x_{10} = 91\\ &(210)\; x_5 + x_6 + x_7 + x_8 + x_9 + x_{10} = 100 \end{aligned} \tag{11}$$

Each combination of ten things taken six at a time yielded one of his equations. His purpose was evidently to demonstrate that over-determined equations can still have a unique solution — which in this case was $(x_1, x_2, \ldots, x_{10}) = (1, 4, 9, 16, 25, 10, 15, 20, 25, 5)$. [Salah Ahmad and Roshdi Rashed, *Al-Bāhir en Algèbre d'As-Samaw'al* (Damascus: 1972), 77–82, ٢٤٨–٢٣١.]

Rolling dice. Some glimmerings of elementary combinatorics arose also in medieval Europe, especially in connection with the question of listing all possible outcomes when three dice are thrown. There are, of course, $\binom{8}{3} = 56$ ways to choose 3 things from 6 when repetitions are allowed. Gambling was officially prohibited; yet these 56 ways became rather well known. In about A.D. 965, Bishop Wibold of Cambrai in northern France devised a game called Ludus Clericalis, so that members of the clergy could enjoy rolling dice while remaining pious. His idea was to associate each possible roll with one of 56 virtues, according to the following table:

⚀⚀⚀	love	⚀⚂⚅	perseverance	⚁⚂⚄	hospitality	⚂⚃⚅	mortification
⚀⚀⚁	faith	⚀⚃⚃	kindness	⚁⚂⚅	economy	⚂⚄⚄	innocence
⚀⚀⚂	hope	⚀⚃⚄	modesty	⚁⚃⚃	patience	⚂⚄⚅	contrition
⚀⚀⚃	justice	⚀⚃⚅	resignation	⚁⚃⚄	zeal	⚂⚅⚅	confession
⚀⚀⚄	prudence	⚀⚄⚄	gentleness	⚁⚃⚅	poverty	⚃⚃⚃	maturity
⚀⚀⚅	temperance	⚀⚄⚅	generosity	⚁⚄⚄	softness	⚃⚃⚄	solicitude
⚀⚁⚁	courage	⚀⚅⚅	wisdom	⚁⚄⚅	virginity	⚃⚃⚅	constancy
⚀⚁⚂	peace	⚁⚁⚁	remorse	⚁⚅⚅	respect	⚃⚄⚄	intelligence
⚀⚁⚃	chastity	⚁⚁⚂	joy	⚂⚂⚂	piety	⚃⚄⚅	sighing
⚀⚁⚄	mercy	⚁⚁⚃	sobriety	⚂⚂⚃	indulgence	⚃⚅⚅	weeping
⚀⚁⚅	obedience	⚁⚁⚄	satisfaction	⚂⚂⚄	prayer	⚄⚄⚄	cheerfulness
⚀⚂⚂	fear	⚁⚁⚅	sweetness	⚂⚂⚅	affection	⚄⚄⚅	compassion
⚀⚂⚃	foresight	⚁⚂⚂	cleverness	⚂⚃⚃	judgment	⚄⚅⚅	self-control
⚀⚂⚄	discretion	⚁⚂⚃	simplicity	⚂⚃⚄	vigilance	⚅⚅⚅	humility

Players took turns, and the first to roll each virtue acquired it. After all possibilities had arisen, the most virtuous player won. Wibold noted that love (*caritas*) is the best virtue of all. He gave a complicated scoring system by which two virtues could be combined if the sum of pips on all six of their dice was 21; for example, love + humility or chastity + intelligence could be paired in this way, and such combinations ranked above any individual virtue. He also considered more complex variants of the game in which vowels appeared on the dice instead of spots, so that virtues could be claimed if their vowels were thrown.

Wibold's table of virtues was presented in lexicographic order, as above, when it was first described by Baldéric in his *Chronicon Cameracense*, about 150 years later. [*Patrologia Latina* **134** (Paris: 1884), 1007–1016.] But another medieval manuscript presented the possible dice rolls in quite a different order:

⚅⚅⚅	⚄⚄⚄	⚃⚃⚃	⚂⚂⚂	⚁⚁⚁	⚀⚀⚀	⚅⚅⚄
⚅⚅⚃	⚅⚅⚂	⚅⚅⚁	⚅⚅⚀	⚄⚄⚅	⚄⚄⚃	⚄⚄⚂
⚄⚄⚁	⚄⚄⚀	⚃⚃⚅	⚃⚃⚄	⚃⚃⚂	⚃⚃⚁	⚃⚃⚀
⚂⚂⚅	⚂⚂⚄	⚂⚂⚃	⚂⚂⚁	⚂⚂⚀	⚁⚁⚅	⚁⚁⚄
⚁⚁⚃	⚁⚁⚂	⚁⚁⚀	⚀⚀⚅	⚀⚀⚄	⚀⚀⚃	⚀⚀⚂
⚀⚀⚁	⚅⚄⚃	⚄⚃⚂	⚃⚂⚁	⚂⚁⚀	⚅⚃⚂	⚅⚃⚀
⚅⚂⚀	⚄⚂⚀	⚅⚄⚂	⚅⚄⚁	⚅⚄⚀	⚅⚁⚀	⚄⚁⚀
⚃⚁⚀	⚄⚃⚁	⚄⚃⚀	⚅⚃⚂	⚅⚃⚁	⚄⚂⚁	⚃⚂⚀

(12)

In this case the author knew how to deal with repeated values, but had no systematic way to handle the cases in which all dice were different; so he listed ⚅⚃⚂ twice and forgot ⚅⚂⚁. [See M. G. Kendall, *Biometrika* **43** (1956), 1–14.]

An amusing poem entitled "Chaunce of the Dyse," attributed to John Lydgate, was written in the early 1400s for use at parties. Its opening verses invite each person to throw three dice; then the remaining verses, which are indexed in decreasing lexicographic order from ⚅⚅⚅ to ⚅⚅⚄ to $\cdots$ to ⚀⚀⚀, give 56 character sketches that light-heartedly describe the thrower. [The full text was published by E. P. Hammond in *Englische Studien* **59** (1925), 1–16; a translation into modern English would be desirable.]

I pray to god that euery wight may caste
Vpon three dyse ryght as is in hys herte
Whether he be rechelesse or stedfaste
So moote he lawghen outher elles smerte
He that is gilty his lyfe to converte
They that in trouthe haue suffred many a throwe
Moote ther chaunce fal as they moote be knowe.

— *The Chaunce of the Dyse* (c. 1410)

Ramon Llull. Significant ripples of combinatorial concepts also emanated from an energetic and quixotic Catalan poet, novelist, encyclopedist, educator, mystic, and missionary named Ramon Llull (c. 1232–1316). Llull's approach to knowledge was essentially to identify basic principles and then to contemplate combining them in all possible ways.

For example, one chapter in his *Ars Compendiosa Inveniendi Veritatem* (c. 1274) began by enumerating sixteen attributes of God: Goodness, greatness, eternity, power, wisdom, love, virtue, truth, glory, perfection, justice, generosity, mercy, humility, sovereignty, and patience. Then Llull wrote $\binom{16}{2} = 120$ short essays of about 80 words each, considering God's goodness as related to greatness, God's goodness re eternity, and so on, ending with God's sovereignty as related to patience. In another chapter he considered seven virtues (faith, hope, charity, justice, prudence, fortitude, temperance) and seven vices (gluttony, lust, greed, sloth, pride, envy, anger), with $\binom{14}{2} = 91$ subchapters to deal with each pair in turn. Other chapters were systematically divided in a similar way, into $\binom{8}{2} = 28$, $\binom{15}{2} = 105$, $\binom{4}{2} = 6$, and $\binom{16}{2} = 120$ subsections. (One wonders what might have happened if he had been familiar with Wibold's list of 56 virtues; would he have produced commentaries on all $\binom{56}{2} = 1540$ of their pairs?)

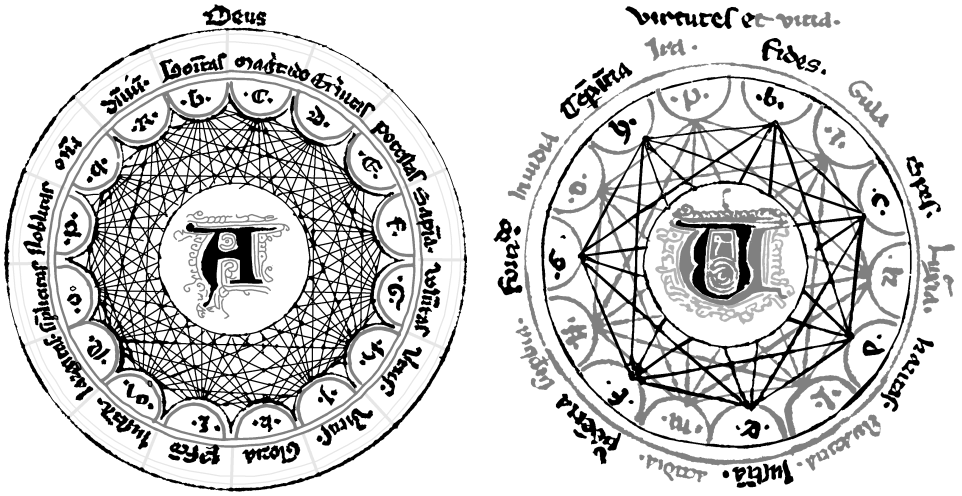
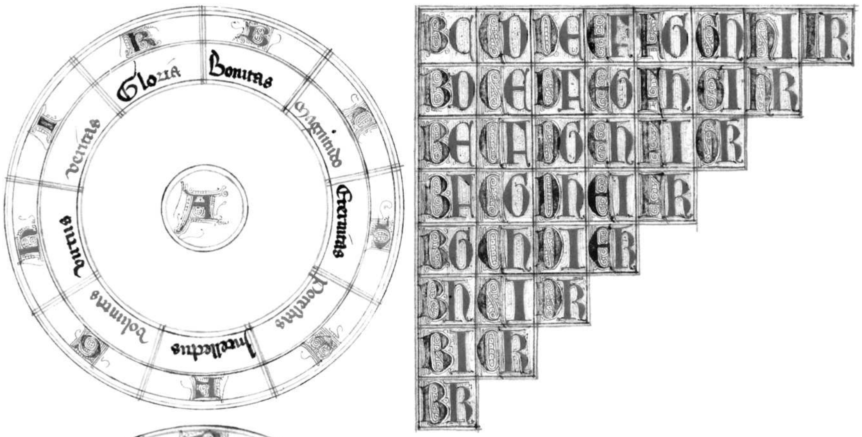


Fig. 44. Illustrations in a manuscript presented by Ramon Llull to the doge of Venice in 1280. [From his *Ars Demonstrativa*, Biblioteca Marciana, VI 200, folio 3^{v}.]

Llull illustrated his methodology by drawing circular diagrams like those in Figure 44. The left-hand circle in this illustration, *Deus*, names sixteen divine attributes — essentially the same sixteen listed earlier, except that love (*amor*) was now called will (*voluntas*), and the final four were now respectively simplicity, rank, mercy, and sovereignty. Each attribute is assigned a code letter, and the illustration depicts their interrelations as the complete graph K_{16} on vertices $(\mathrm{B, C, D, E, F, G, H, I, K, L, M, N, O, P, Q, R})$. The right-hand figure, *virtutes et vitia*, shows the seven virtues (b, c, d, e, f, g, h) interleaved with the seven vices (i, k, l, m, n, o, p); in the original manuscript virtues appeared in blue ink while vices appeared in red. Notice that in this case his illustration depicted two independent complete graphs K_7, one of each color. (He no longer bothered to compare each individual virtue with each individual vice, since every virtue was clearly better than every vice.)

Llull used the same approach to write about medicine: Instead of juxtaposing theological concepts, his *Liber Principiorum Medicinæ* (c. 1275) considered combinations of symptoms and treatments. And he also wrote books



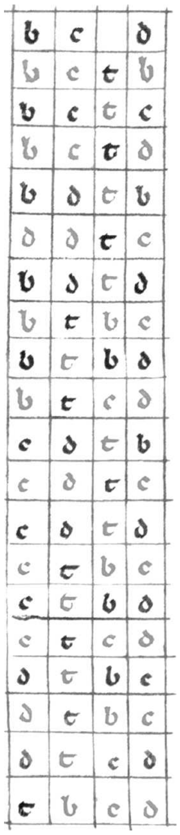


Fig. 45. Llullian illustrations from a manuscript presented to the queen of France, c. 1325. [Badische Landesbibliothek Karlsruhe, Codex St. Peter perg. 92, folios 28^{v} and 39^{v}.]

on philosophy, logic, jurisprudence, astrology, zoology, geometry, rhetoric, and chivalry — more than 200 works in all. It must be admitted, however, that much of this material was highly repetitive; modern data compression techniques would probably reduce Llull's output to a size much less than that of, say, Aristotle.

He eventually decided to simplify his system by working primarily with groups of nine things. See, for example, Fig. 45, where circle A now lists only the first nine of God's attributes $(\mathrm{B, C, D, E, F, G, H, I, K})$. The $\binom{9}{2} = 36$ associated pairs $(\mathrm{BC, BD}, \ldots, \mathrm{IK})$ appear in the stairstep chart at the right of that circle. By adding two more virtues, namely patience and compassion — as well as two more vices, namely lying and inconsistency — he could treat virtues vis-à-vis virtues and vices vis-à-vis vices with the same chart. He also proposed using the same chart to carry out an interesting scheme for voting, in an election with nine candidates [see I. McLean and J. London, *Studia Lulliana* **32** (1992), 21–37].

The encircled triangles at the lower left of Fig. 45 illustrate another key aspect of Llull's approach. Triangle $(\mathrm{B, C, D})$ stands for (difference, concordance, contrariness); triangle $(\mathrm{E, F, G})$ stands for (beginning, middle, ending); and triangle $(\mathrm{H, I, K})$ stands for (greater, equal, less). These three interleaved appearances of K_3 represent three kinds of three-valued logic. Llull had experimented earlier with other such triplets, notably '(true, unknown, false)'. We can get an idea of how he used the triangles by considering how he dealt with combinations of the four basic elements (earth, air, fire, water): All four elements are different; earth is concordant with fire, which concords with air, which concords with water, which concords with earth; earth is contrary to air, and fire is contrary to water; these considerations complete an analysis with respect to triangle (B, C, D). Turning to triangle (E, F, G), he noted that various processes in nature begin with one element dominating another; then a transition or middle state occurs, until a goal is reached, like air becoming warm. For triangle (H, I, K) he said that in general we have fire > air > water > earth with respect to their "spheres," their "velocities," and their "nobilities"; nevertheless we also have, for example, air > fire with respect to supporting life, while air and fire have equal value when they are working together.

Llull provided the vertical table at the right of Fig. 45 as a further aid. (See exercise 11 below.) He also introduced movable concentric wheels, labeled with the letters (B, C, D, E, F, G, H, I, K) and with other names, so that many things could be contemplated simultaneously. In this way a faithful practitioner of the Llullian art could be sure to have all the bases covered.

Several centuries later, Athanasius Kircher published an extension of Llull's system as part of a large tome entitled *Ars Magna Sciendi sive Combinatoria* (Amsterdam: 1669), with five movable wheels accompanying page 173 of that book. Kircher also extended Llull's repertoire of complete graphs K_n by providing illustrations of complete *bipartite* graphs $K_{m,n}$; for example, Fig. 46 is taken from page 171 of Kircher's book, and his page 170 contains a glorious picture of $K_{18,18}$.

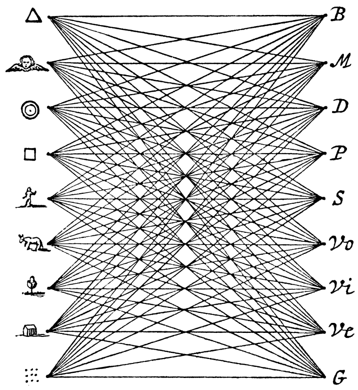


Fig. 46. $K_{9,9}$ as presented by Athanasius Kircher in 1669.

It is an investigative and inventive art.
When ideas are combined in all possible ways,
the new combinations start the mind thinking along novel channels
and one is led to discover fresh truths and arguments.
— MARTIN GARDNER, *Logic Machines and Diagrams* (1958)

The most extensive modern development of Llull-like methods is perhaps *The Schillinger System of Musical Composition* by Joseph Schillinger (New York:

Carl Fischer, 1946), a remarkable two-volume work that presents theories of rhythm, melody, harmony, counterpoint, composition, orchestration, etc., from a combinatorial perspective. On page 56, for example, Schillinger lists the 24 permutations of $\{a,b,c,d\}$ in the Gray-code order of plain changes (Algorithm 7.2.1.2P); then on page 57 he applies them not to pitches but rather to rhythms, to the durations of notes. On page 364 he exhibits the symmetrical cycle

$$(2, 0, 3, 4, 2, 5, 6, 4, 0, 1, 6, 2, 3, 1, 4, 5, 3, 6, 0, 5, 1), \tag{13}$$

a universal cycle of 2-combinations for the seven objects $\{0,1,2,3,4,5,6\}$; in other words, (13) is an Eulerian trail in K_7: All $\binom{7}{2} = 21$ pairs of digits occur exactly once. Such patterns are grist for a composer's mill. But we can be grateful that Schillinger's better students (like George Gershwin) did not commit themselves entirely to a strictly mathematical sense of aesthetics.

Tacquet, van Schooten, and Izquierdo. Three additional books related to our story were published during the 1650s. André Tacquet wrote a popular text, *Arithmeticæ Theoria et Praxis* (Louvain: 1656), that was reprinted and revised often during the next fifty years. Near the end, on pages 376 and 377, he gave a procedure for listing combinations two at a time, then three at a time, etc.

Frans van Schooten's *Exercitationes Mathematicæ* (Leiden: 1657) was more advanced. On page 373 he listed all combinations in an appealing layout

$$\begin{array}{c} \underline{a} \\ \underline{b.\ ab} \\ \underline{c.\ ac.\ bc.\ abc} \\ d.\ ad.\ bd.\ abd.\ cd.\ acd.\ bcd.\ abcd \end{array} \tag{14}$$

and he proceeded on the next few pages to extend this pattern to the letters e, f, g, h, i, k, "et sic in infinitum." On page 376 he observed that one can replace (a,b,c,d) by $(2,3,5,7)$ in (14) to get the divisors of 210 that exceed unity:

$$\begin{array}{c} \underline{2} \\ \underline{3\ \ 6} \\ \underline{5\ \ 10\ \ 15\ \ 30} \\ 7\ \ 14\ \ 21\ \ 42\ \ 35\ \ 70\ \ 105\ \ 210 \end{array} \tag{15}$$

And on the following page he extended the idea to

$$\begin{array}{c} \underline{a} \\ \underline{a.\ aa} \\ \underline{b.\ ab.\ aab} \\ c.\ ac.\ aac.\ bc.\ abc.\ aabc \end{array} \tag{16}$$

thereby allowing two a's. He didn't really understand this extension, though; his next example

$$\begin{array}{c} \underline{a} \\ \underline{a.\ aa} \\ \underline{a.\ aaa} \\ \underline{b.\ ab.\ aab.\ aaab} \\ b.\ bb.\ abb.\ aabb.\ aaabb \end{array} \tag{17}$$

was botched, indicating the limits of his knowledge at the time. (See exercise 13.)

On page 411 van Schooten observed that the weights $(a, b, c, d) = (1, 2, 4, 8)$ could be assigned in (14), leading to

$$\begin{array}{c} \underline{1} \\ \underline{2\ \ 3} \\ \underline{4\ \ 5\ \ 6\ \ 7} \\ 8\ \ 9\ \ 10\ \ 11\ \ 12\ \ 13\ \ 14\ \ 15 \end{array} \tag{18}$$

after addition. But he didn't see the connection with radix-2 arithmetic.

Sebastián Izquierdo's two-volume work *Pharus Scientiarum* (Lyon: 1659), "The Lighthouse of Science," included a nicely organized discussion of combinatorics entitled Disputatio 29, *De Combinatione*. He gave a detailed discussion of four key parts of Stanley's Twelvefold Way, namely the n-tuples, n-variations, n-multicombinations, and n-combinations of m objects that appear in the first two rows and the first two columns of Table 7.2.1.4–1.

In Sections 81–84 of *De Combinatione* he listed all combinations of m letters taken n at a time, for $2 \le n \le 5$ and $n \le m \le 9$, always in lexicographic order; he also tabulated them for $m = 10$ and 20 in the cases $n = 2$ and 3. But when he listed the $m^{\underline{n}}$ *variations* of m things taken n at a time, he chose a more complicated ordering (see exercise 14).

Izquierdo was first to discover the formula $\binom{m+n-1}{n}$ for combinations of m things taken n at a time with unlimited repetition; this rule appeared in §48–§51 of his work. But in §105, when he attempted to list all such combinations in the case $n = 3$, he didn't know that there was a simple way to do it. In fact, his listing of the 56 cases for $m = 6$ was rather like the old, awkward ordering of (12).

Combinations with repetition were not well understood until James Bernoulli's *Ars Conjectandi*, "The Art of Guessing," came out in 1713. In Part 2, Chapter 5, Bernoulli simply listed the possibilities in lexicographic order, and showed that the formula $\binom{m+n-1}{n}$ follows by induction as an easy consequence. [Niccolò Tartaglia had, incidentally, come close to discovering this formula in his *General trattato di numeri, et misure* **2** (Venice: 1556), 17^{r} and 69^{v}; so had the Maghrebi mathematician Ibn Mun'im in his 13th-century *Fiqh al-Ḥisāb*.]

The null case. Before we conclude our discussion of early work on combinations, we should not forget a small yet noble step taken by John Wallis on page 110 of his *Discourse of Combinations* (1685), where he specifically considered the combination of m things taken 0 at a time: "It is manifest, That, if we would *take None*, that is, if would *leave All*; there can be but one case thereof, what ever be the Number of things exposed." Furthermore, on page 113, he knew that $\binom{0}{0} = 1$: "(for, here, to take all, or to leave all, is but one and the same case.)"

However, when he gave a table of $n!$ for $n \le 24$, he did not go so far as to point out that $0! = 1$, or that there is exactly one permutation of the empty set.

The work of Nārāyaṇa. A remarkable monograph entitled *Gaṇita Kaumudī* ("Treatise on Calculation"), written by Nārāyaṇa Paṇḍita in 1356, has recently become known in detail to scholars outside of India for the first time, thanks to an English translation by Parmanand Singh [*Gaṇita Bhāratī* **20** (1998), 25–82; **21** (1999), 10–73; **22** (2000), 19–85; **23** (2001), 18–82; **24** (2002), 35–98].

Chapter 13 of his work, subtitled *Aṅka Pāśa* ("Concatenation of Numbers"), was devoted to combinatorial generation. Indeed, although the 97 "sutras" of this chapter were rather cryptic, they presented a comprehensive theory of the subject that anticipated developments in the rest of the world by several hundred years.

For example, Nārāyaṇa dealt with permutation generation in sutras 49–55a, where he gave algorithms to list all permutations of a set in decreasing colex order, together with algorithms to rank a given permutation and to unrank a given serial number. In this way he essentially discovered the factorial representation of positive integers. Then in sutras 57–60 he extended the algorithms to handle general multisets; for example, he listed the permutations of $\{1,1,2,4\}$ as

$$1124, 1214, 2114, 1142, 1412, 4112, 1241, 2141, 1421, 4121, 2411, 4211,$$

again in decreasing colex order.

Nārāyaṇa's sutras 88–92 dealt with systematic generation of combinations. Besides illustrating the combinations of $\{1,\ldots,8\}$ taken 3 at a time, namely

$$(678,\ 578,\ 478,\ \ldots,\ 134,\ 124,\ 123),$$

he also considered a bit-string representation of these combinations in the reverse order (*increasing* colex):

$$(11100000,\ 11010000,\ 10110000,\ \ldots, 00010011,\ 00001011,\ 00000111).$$

He almost, but not quite, discovered Theorem 7.2.1.3L.

Thus we can legitimately regard Nārāyaṇa Paṇḍita as the founder of the science of combinatorial generation — even though, like many other pioneers who were significantly "ahead of their time," his work on the subject never became well known even in his own country.

Permutable poetry. Let's turn now to a curious question that attracted the attention of several prominent mathematicians in the seventeenth century, because it sheds considerable light on the state of combinatorial knowledge in Europe at that time. A Jesuit priest named Bernard Bauhuis had composed a famous one-line tribute to the Virgin Mary, in Latin hexameter:

Tot tibi sunt dotes, Virgo, quot sidera cælo. (19)

["Thou hast as many virtues, O Virgin, as there are stars in heaven"; see his *Epigrammatum Libri V* (Cologne: 1615), 49.] His verse inspired Erycius Puteanus, a professor at the University of Louvain, to write a book entitled *Pietatis Thaumata* (Antwerp: 1617), presenting 1022 permutations of Bauhuis's words. For example, Puteanus wrote

107 Tot dotes tibi, quot cælo sunt sidera, Virgo.
270 Dotes tot, cælo sunt sidera quot, tibi Virgo.
329 Dotes, cælo sunt quot sidera, Virgo tibi tot.
384 Sidera quot cælo, tot sunt Virgo tibi dotes. (20)
725 Quot cælo sunt sidera, tot Virgo tibi dotes.
949 Sunt dotes Virgo, quot sidera, tot tibi cælo.
1022 Sunt cælo tot Virgo tibi, quot sidera, dotes.

He stopped at 1022, because 1022 was the number of visible stars in Ptolemy's well-known catalog of the heavens.

The idea of permuting words in this way was well known at the time; such wordplay was what Julius Scaliger had called "Proteus verses" in his *Poetices Libri Septem* (Lyon: 1561), Book 2, Chapter 30. The Latin language lends itself to permutations like (20), because Latin word endings tend to define the function of each noun, making the relative word order much less important to the meaning of a sentence than it is in English. Puteanus did state, however, that he had specifically avoided unsuitable permutations such as

$$\text{Sidera tot cælo, Virgo, quot sunt tibi dotes,} \tag{21}$$

because they would place an *upper* bound on the Virgin's virtues rather than a lower bound. [See pages 12 and 103 of his book.]

Of course there are $8! = 40{,}320$ ways to permute the words of (19). But that wasn't the point; most of those ways don't "scan." Each of Puteanus's 1022 verses obeyed the strict rules of classical *hexameter*, the rules that had been followed by Greek and Latin poets since the days of Homer and Vergil, namely:

i) Each word consists of syllables that are either long ($-$) or short ($\smile$);
ii) The syllables of each line belong to one of 32 patterns,

$$\left\{\begin{matrix} -\smile\smile \\ -\,- \end{matrix}\right\} \left\{\begin{matrix} -\smile\smile \\ -\,- \end{matrix}\right\} \left\{\begin{matrix} -\smile\smile \\ -\,- \end{matrix}\right\} \left\{\begin{matrix} -\smile\smile \\ -\,- \end{matrix}\right\} -\smile\smile \left\{\begin{matrix} -\smile \\ -\,- \end{matrix}\right\}. \tag{22}$$

In other words there are six metrical feet, where each of the first four is either a dactyl or a spondee in the terminology of (5); the fifth foot should be a dactyl, and the last is either trochee or spondee.

The rules for long versus short syllables in Latin poetry are somewhat tricky in general, but the eight words of Bauhuis's verse can be characterized by the following patterns:

$$\text{tot} = -,\ \ \text{tibi} = \left\{\begin{matrix} \smile\smile \\ \smile\,- \end{matrix}\right\},\ \ \text{sunt} = -,\ \ \text{dotes} = -\,-,$$
$$\text{Virgo} = \left\{\begin{matrix} -\smile \\ -\,- \end{matrix}\right\},\ \ \text{quot} = -,\ \ \text{sidera} = -\smile\smile,\ \ \text{cælo} = -\,-. \tag{23}$$

Notice that poets had two choices when they used the words 'tibi' or 'Virgo'. Thus, for example, (19) fits the hexameter pattern

$$\begin{array}{cccccc} -\ \smile\smile & -\ \ - & -\ \ - & -\ \ - & -\smile\smile & -\ - \\ \text{Tot ti-bi} & \text{sunt do-} & \text{tes, Vir-} & \text{go, quot} & \text{si-de-ra} & \text{cæ-lo.} \end{array} \tag{24}$$

(Dactyl, spondee, spondee, spondee, dactyl, spondee; "dum-diddy dum-dum dum-dum dum-dum dum-diddy dum-dum." The commas represent slight pauses, called "cæsuras," when the words are read; they don't concern us here, although Puteanus inserted them carefully into each of his 1022 permutations.)

A natural question now arises: If we permute Bauhuis's words at random, what are the odds that they scan? In other words, how many of the permutations obey rules (i) and (ii), given the syllable patterns in (23)? G. W. Leibniz raised this question, among others, in his *Dissertatio de Arte Combinatoria* (1666), a work published when he was applying for a position at the University of Leipzig. At this time Leibniz was just 19 years old, largely self-taught, and his understanding of combinatorics was quite limited; for example, he believed that there are 600 permutations of $\{\mathrm{ut}, \mathrm{ut}, \mathrm{re}, \mathrm{mi}, \mathrm{fa}, \mathrm{sol}\}$ and 480 of $\{\mathrm{ut}, \mathrm{ut}, \mathrm{re}, \mathrm{re}, \mathrm{mi}, \mathrm{fa}\}$, and he even stated that (22) represents 76 possibilities instead of 32. [See §5 and §8 in his Problem 6.]

But Leibniz did realize that it would be worthwhile to develop general methods for counting all permutations that are "useful," in situations when many permutations are "useless." He considered several examples of Proteus verses, enumerating some of the simpler ones correctly but making many errors when the words were complicated. Although he mentioned Puteanus's work, he didn't attempt to enumerate the scannable permutations of (19).

A much more successful approach was introduced a few years later by Jean Prestet in his *Élémens des Mathématiques* (Paris: 1675), 342–438. Prestet gave a clear exposition leading to the conclusion that exactly 2196 permutations of Bauhuis's verse would yield a proper hexameter. However, he soon realized that he had forgotten to count quite a few cases — including those numbered 270, 384, and 725 in (20). So he completely rewrote this material when he published *Nouveaux Élémens des Mathématiques* in 1689. Pages 127–133 of Prestet's new book were devoted to showing that the true number of scannable permutations was 3276, almost 50% larger than his previous total.

Meanwhile John Wallis had treated the problem in his *Discourse of Combinations* (London: 1685), 118–119, published as a supplement to his *Treatise of Algebra*. After explaining why he believed the correct number to be 3096, Wallis admitted that he may have overlooked some possibilities and/or counted some cases more than once; "but I do not, at present, discern either the one and other."

An anonymous reviewer of Wallis's work remarked that the true number of metrically correct permutations was actually 2580 — but he gave no proof [*Acta Eruditorum* **5** (1686), 289]. The reviewer was almost certainly G. W. Leibniz himself, although no clue to the reasoning behind the number 2580 has been found among Leibniz's voluminous unpublished notes.

Finally James Bernoulli entered the picture. In his inaugural lecture as Dean of Philosophy at the University of Basel, 1692, he mentioned the tot-tibi enumeration problem and stated that a careful analysis is necessary to obtain the correct answer — which, he said, was 3312(!). His proof appeared posthumously in the first edition of his *Ars Conjectandi* (1713), 79–81. [Those pages were, incidentally, omitted from later editions of that famous book, and from his collected works, because he didn't actually intend them for publication; a proofreader had inserted them by mistake. See *Die Werke von Jakob Bernoulli* **3** (Basel: Birkhäuser, 1975), 78, 98–106, 108, 154–155.]

So who was right? Are there 2196 scannable permutations, or 3276, or 3096, or 2580, or 3312? W. A. Whitworth and W. E. Hartley considered the question anew in *The Mathematical Gazette* **2** (1902), 227–228, where they each presented elegant arguments and concluded that the true total was in fact none of the above. Their joint answer, 2880, represented the first time that any two mathematicians had independently come to the same conclusion about this problem.

But exercises 20 and 21, below, reveal the truth: Bernoulli is vindicated, and everybody else was wrong. Moreover, a study of Bernoulli's systematic and carefully indented 3-page derivation indicates that he was successful chiefly because he adhered faithfully to a discipline that we now call the *backtrack method.* We shall study the backtrack method thoroughly in Section 7.2.2, where we will also see that the tot-tibi question is readily solved as a special case of the *exact cover problem.*

> *Even the wisest and most prudent people often suffer from what Logicians call* insufficient enumeration of cases.
>
> — JAMES BERNOULLI (1692)

Set partitions. The partitions of a set seem to have been studied first in Japan, where a parlor game called *genji-ko* ("Genji incense") became popular among upperclass people about A.D. 1500. The host of a gathering would secretly select five packets of incense, some of which might be identical, and he would burn them one at a time. The guests would try to discern which of the scents were the same and which were different; in other words, they would try to guess which of the $\varpi_5 = 52$ partitions of $\{1, 2, 3, 4, 5\}$ had been chosen by their host.

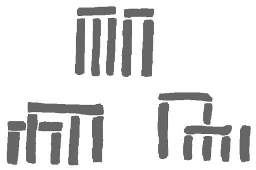
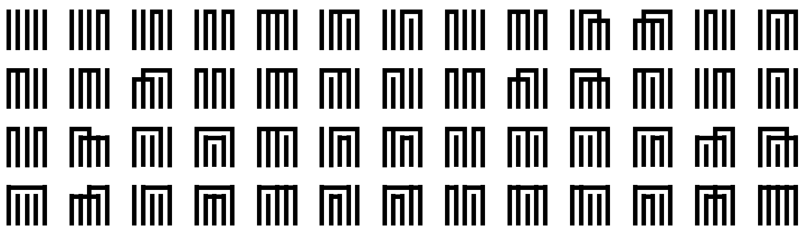


Fig. 47. Diagrams used to represent set partitions in 16th century Japan. [From a copy in the collection of Tamaki Yano at Saitama University.]

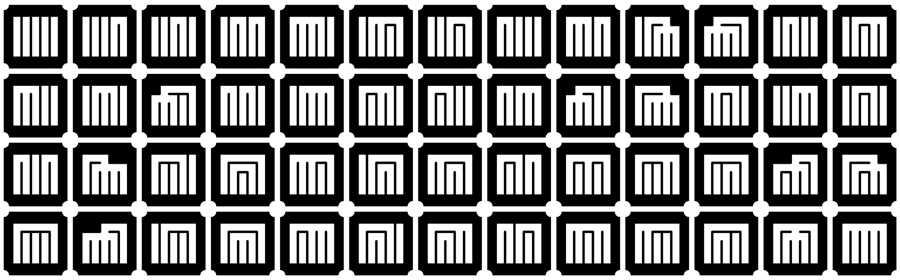
Soon it became customary to represent the 52 possible outcomes by diagrams like those in Fig. 47. For example, the uppermost diagram of that illustration, when read from right to left, would indicate that the first two scents are identical and so are the last three; thus the partition is $12|345$. The other two diagrams, similarly, are pictorial ways to represent the respective partitions $124|35$ and $1|24|35$. As an aid to memory, each of the 52 patterns was named after a chapter of Lady Murasaki's famous 11th-century *Tale of Genji*, according to the following sequence [*Encyclopedia Japonicæ* (Tokyo: Sanseido, 1910), 1299]:



(25)

(Once again, as we've seen in many other examples, the possibilities were not arranged in any particularly logical order.)

The appealing nature of these genji-ko patterns led many families to adopt them as heraldic crests. For example, the following stylized variants of (25) were found in standard catalogs of kimono patterns early in the 20th century:



[See Fumie Adachi, *Japanese Design Motifs* (New York: Dover, 1972), 150–153.]

Early in the 1700s, Takakazu Seki and his students began to investigate the number of set partitions ϖ_n for arbitrary n, inspired by the known result that $\varpi_5 = 52$. Yoshisuke Matsunaga found formulas for the number of set partitions when there are k_j subsets of size n_j for $1 \le j \le t$, with $k_1 n_1 + \cdots + k_t n_t = n$ (see the answer to exercise 1.2.5–21). He also discovered the basic recurrence relation 7.2.1.5–(14), namely

$$\varpi_{n+1} \;=\; \binom{n}{0}\varpi_n + \binom{n}{1}\varpi_{n-1} + \binom{n}{2}\varpi_{n-2} + \cdots + \binom{n}{n}\varpi_0, \tag{26}$$

by which the values of ϖ_n can readily be computed.

Matsunaga's discoveries remained unpublished until Yoriyuki Arima's book *Shūki Sanpō* came out in 1769. Problem 56 of that book asked the reader to solve the equation "$\varpi_n = 678570$" for n; and Arima's answer, worked out in detail (with credit duly given to Matsunaga), was $n = 11$.

Shortly afterwards, Masanobu Saka studied the number $\left\{ {n \atop k} \right\}$ of ways that an n-set can be partitioned into k subsets, in his work *Sanpō-Gakkai* (1782). He discovered the recurrence formula

$$\left\{ {n+1 \atop k} \right\} \;=\; k\left\{ {n \atop k} \right\} + \left\{ {n \atop k-1} \right\}, \tag{27}$$

and tabulated the results for $n \le 11$. James Stirling, in his *Methodus Differentialis* (1730), had discovered the numbers $\left\{ {n \atop k} \right\}$ in a purely algebraic context; thus Saka was the first person to realize their combinatorial significance.

An interesting algorithm for listing set partitions was subsequently devised by Toshiaki Honda (see exercise 23). Further details about genji-ko and its relation to the history of mathematics can be found in Japanese articles by Tamaki Yano, *Sugaku Seminar* **34**, 11 (Nov. 1995), 58–61; **34**, 12 (Dec. 1995), 56–60.

Set partitions remained virtually unknown in Europe until much later, except for three isolated incidents. First, George and/or Richard Puttenham published *The Arte of English Poesie* in 1589, and pages 70–72 of that book contain diagrams similar to those of genji-ko. For example, the seven diagrams

(28)

were used to illustrate possible rhyme schemes for 5-line poems, "whereof some of them be harsher and unpleasaunter to the eare then other some be." But this visually appealing list was incomplete (see exercise 24).

Second, an unpublished manuscript of G. W. Leibniz from the late 1600s shows that he had tried to count the number of ways to partition $\{1,\ldots,n\}$ into three or four subsets, but with almost no success. He enumerated ${n \brace 2}$ by a very cumbersome method, which would not have led him to see readily that ${n \brace 2} = 2^{n-1} - 1$. He attempted to compute ${n \brace 3}$ and ${n \brace 4}$ only for $n \le 5$, and made several numerical slips leading to incorrect answers. [See E. Knobloch, *Studia Leibnitiana Supplementa* **11** (1973), 229–233; **16** (1976), 316–321.]

The third European appearance of set partitions had a completely different character. John Wallis devoted the third chapter of his *Discourse of Combinations* (1685) to questions about "aliquot parts," the proper divisors of numbers, and in particular he studied the set of all ways to factorize a given integer. This question is equivalent to the study of *multiset* partitions; for example, the factorizations of p^3q^2r are essentially the same as the partitions of $\{p,p,p,q,q,r\}$, when p, q, and r are prime numbers. Wallis devised an excellent algorithm for listing all factorizations of a given integer n, essentially anticipating Algorithm 7.2.1.5M (see exercise 27). But he didn't investigate the important special cases that arise when n is the power of a prime (equivalent to integer partitions) or when n is squarefree (equivalent to set partitions). Thus, although Wallis was able to solve the more general problem, its complexities paradoxically deflected him from discovering partition numbers, Bell numbers, or Stirling subset numbers, or from devising simple algorithms that would generate integer partitions or set partitions.

Integer partitions. Partitions of integers arrived on the scene even more slowly. Bishop Wibold (c. 965) knew the partitions of n into exactly three parts ≤ 6. So did Galileo, who wrote a memo about them (c. 1627) and also studied their frequency of occurrence as rolls of three dice. ["Sopra le scoperte de i dadi," in Galileo's *Opere*, Volume 8, 591–594; he listed partitions in decreasing lexicographic order.]

Mersenne listed the partitions of 9 into any number of parts, on page 130 of his *Traitez de la Voix et des Chants* (1636). With each partition $9 = a_1 + \cdots + a_k$ he also computed the multinomial coefficient $9!/(a_1! \ldots a_k!)$; as we've seen earlier, he was interested in counting various melodies, and he knew for example that there are $9!/(3!3!3!) = 1680$ melodies on the nine notes $\{a,a,a,b,b,b,c,c,c\}$. But he failed to mention the cases $8+1$ and $3+2+1+1+1+1$, probably because he hadn't listed the possibilities in any systematic way.

Leibniz considered two-part partitions in Problem 3 of his *Dissertatio de Arte Combinatoria* (1666), and his unpublished notes show that he subsequently spent considerable time trying to enumerate the partitions that have three or more summands. He called them "discerptions," or (less frequently) "divulsions" — in Latin of course — or sometimes "sections" or "dispersions" or even "partitions." He was interested in them primarily because of their connection with the monomial symmetric functions $\sum x_{i_1}^{a_1} x_{i_2}^{a_2} \ldots$. But his many attempts led to almost total failure, except in the case of three summands, when he almost (but not quite) discovered the formula for $\left|{n \atop 3}\right|$ in exercise 7.2.1.4–31. For example, he carelessly counted only 21 partitions of 8, forgetting the case $2+2+2+1+1$; and he got only 26 for $p(9)$, after missing $3+2+2+2$, $3+2+2+1+1$, $2+2+2+1+1+1$, and $2+2+1+1+1+1+1$ — in spite of the fact that he was trying to list partitions systematically in decreasing lexicographic order. [See E. Knobloch, *Studia Leibnitiana Supplementa* **11** (1973), 91–258; **16** (1976), 255–337; *Historia Mathematica* **1** (1974), 409–430.]

Abraham de Moivre had the first real success with partitions, in his paper "A Method of Raising an infinite Multinomial to any given Power, or Extracting any given Root of the same" [*Philosophical Transactions* **19** (1697), 619–625 and Fig. 5]. He proved that the coefficient of z^{m+n} in $(az+bz^2+cz^3+\cdots)^m$ has one term for each partition of n; for example, the coefficient of z^{m+6} is

$$\begin{aligned}\tbinom{m}{6}a^{m-6}b^6 + 5\tbinom{m}{5}a^{m-5}b^4c + 4\tbinom{m}{4}a^{m-4}b^3d + 6\tbinom{m}{4}a^{m-4}b^2c^2\\ + 3\tbinom{m}{3}a^{m-3}b^2e + 6\tbinom{m}{3}a^{m-3}bcd + 2\tbinom{m}{2}a^{m-2}bf + \tbinom{m}{3}a^{m-3}c^3\\ + 2\tbinom{m}{2}a^{m-2}ce + \tbinom{m}{2}a^{m-2}d^2 + \tbinom{m}{1}a^{m-1}g. \qquad (29)\end{aligned}$$

If we set $a=1$, the term with exponents $b^ic^jd^ke^l\ldots$ corresponds to the partition with i 1s, j 2s, k 3s, l 4s, etc. Thus, for example, when $n=6$ he essentially presented the partitions in the order

$$111111,\ 11112,\ 1113,\ 1122,\ 114,\ 123,\ 15,\ 222,\ 24,\ 33,\ 6. \qquad (30)$$

He explained how to list the partitions recursively, as follows (but in different language related to his own notation): For $k = 1, 2, \ldots, n$, start with k and append the (previously listed) partitions of $n-k$ whose smallest part is $\geq k$.

[My solution] was ordered to be published in the Transactions,
not so much as a matter relating to Play,
but as containing some general Speculations
not unworthy to be considered by the Lovers of Truth.
— ABRAHAM DE MOIVRE (1717)

P. R. de Montmort tabulated all partitions of numbers ≤ 9 into ≤ 6 parts in his *Essay d'Analyse sur les Jeux de Hazard* (1708), in connection with dice problems. His partitions were listed in a different order from (30); for example,

$$111111,\ 21111,\ 2211,\ 222,\ 3111,\ 321,\ 33,\ 411,\ 42,\ 51,\ 6. \qquad (31)$$

He probably was unaware of de Moivre's prior work.

So far almost none of the authors we've been discussing actually described the procedures by which they generated combinatorial patterns. We can only infer their methods, or lack thereof, by studying the lists that they actually published. Furthermore, in rare cases such as de Moivre's paper where a tabulation method *was* explicitly described, the author assumed that all patterns for the first cases 1, 2, ..., $n-1$ had been listed before it was time to tackle the case of order n. No method for generating patterns "on the fly," moving directly from one pattern to its successor without looking at auxiliary tables, was actually explained by any of the authors we have encountered, except for Kedāra and Nārāyaṇa. Today's computer programmers naturally prefer methods that are more direct and need little memory.

Roger Joseph Boscovich published the first direct algorithm for partition generation in *Giornale de' Letterati* (Rome, 1747), on pages 393–404 together with two foldout tables facing page 404. His method, which produces for $n = 6$ the respective outputs

$$111111,\ 11112,\ 1122,\ 222,\ 1113,\ 123,\ 33,\ 114,\ 24,\ 15,\ 6, \qquad (32)$$

generates partitions in precisely the reverse order from which they are visited by Algorithm 7.2.1.4P; and his method would indeed have been featured in Section 7.2.1.4, except for the fact that the reverse order turns out to be slightly easier and faster than the order that he had chosen.

Boscovich published sequels in *Giornale de' Letterati* (Rome, 1748), 12–27 and 84–99, extending his algorithm in two ways. First, he considered generating only partitions whose parts belong to a given set S, so that symbolic multinomials with sparse coefficients could be raised to the mth power. (He said that the gcd of all elements of S should be 1; in fact, however, his method could fail if $1 \notin S$.) Second, he introduced an algorithm for generating partitions of n into m parts, given m and n. Again he was unlucky: A slightly better way to do that task, Algorithm 7.2.1.4H, was found subsequently, diminishing his chances for fame.

Hindenburg's hype. The inventor of Algorithm 7.2.1.4H was Carl Friedrich Hindenburg, who also rediscovered Nārāyaṇa's Algorithm 7.2.1.2L, a winning technique for generating multiset permutations. Unfortunately, these small successes led him to believe that he had made revolutionary advances in mathematics — although he did condescend to remark that other people such as de Moivre, Euler, and Lambert had come close to making similar discoveries.

Hindenburg was a prototypical overachiever, extremely energetic if not inspired. He founded or cofounded Germany's first professional journals of mathematics (published 1786–1789 and 1794–1800), and contributed long articles to each. He served several times as academic dean at the University of Leipzig, where he was also the Rector in 1792. If he had been a better mathematician, German mathematics might well have flourished more in Leipzig than in Berlin or Göttingen.

But his first mathematical work, *Beschreibung einer ganz neuen Art, nach einem bekannten Gesetze fortgehende Zahlen durch Abzählen oder Abmessen bequem und sicher zu finden* (Leipzig: 1776), amply foreshadowed what was to come: His "ganz neue" (completely new) idea in that booklet was simply to give combinatorial significance to the digits of numbers written in decimal notation. Incredibly, he concluded his monograph with large foldout sheets that contained a table of the numbers 0000 through 9999 — followed by two other tables that listed the even numbers and odd numbers separately(!).

Hindenburg published letters from people who praised his work, and invited them to contribute to his journals. In 1796 he edited *Sammlung combinatorisch-analytischer Abhandlungen*, whose subtitle stated (in German) that de Moivre's multinomial theorem was "the most important proposition in all of mathematical analysis." About a dozen people joined forces to form what became known as Hindenburg's Combinatorial School, and they published thousands of pages filled with esoteric symbolism that must have impressed many nonmathematicians.

The work of this School was not completely trivial from the standpoint of computer science. For example, H. A. Rothe, who was Hindenburg's best student, noticed that there is a simple way to go from a Morse code sequence to its lexicographic successor or predecessor. Another student, J. C. Burkhardt, observed that Morse code sequences of length n could also be generated easily by first considering those with no dashes, then one dash, then two, etc. Their motivation was not to tabulate poetic meters of n beats, as it had been in India, but rather to list the terms of the continuant polynomials $K(x_1, x_2, \ldots, x_n)$, Eq. 4.5.3–(4). [See *Archiv für reine und angewandte Mathematik* **1** (1794), 154–194.] Furthermore, on page 53 of Hindenburg's 1796 *Sammlung* cited above, G. S. Klügel introduced a way to list all permutations that has subsequently become known as Ord-Smith's algorithm; see Eqs. (23)–(26) in Section 7.2.1.2.

Hindenburg believed that his methods deserved equal time with algebra, geometry, and calculus in the standard curriculum. But he and his disciples were combinatorialists who only made combinatorial lists. Burying themselves in formulas and formalisms, they rarely discovered any new mathematics of real interest. Eugen Netto has admirably summarized their work in M. Cantor's *Geschichte der Mathematik* **4** (1908), 201–219. "For a while they controlled the German market; however, most of what they dug up soon sank into a not-entirely-deserved oblivion."

The sad outcome was that combinatorial studies in general got a bad name. Gösta Mittag-Leffler, who assembled a magnificent library of mathematical literature about 100 years after Hindenburg's death, decided to place all such work on a special shelf marked "Dekadenter." And this category still persists in the library of Sweden's Institut Mittag-Leffler today, even as that institute attracts world-class combinatorial mathematicians whose research is anything but decadent.

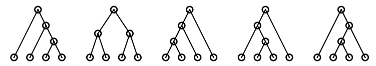
Looking on the bright side, we may note that at least one good book did emerge from all of this activity. Andreas von Ettingshausen's *Die combinatorische Analysis* (Vienna: 1826) is noteworthy as the first text to discuss combinatorial generation methods in a perspicuous way. He discussed the general principles of lexicographic generation in §8, and applied them to construct good ways to list all permutations (§11), combinations (§30), and partitions (§41–§44).

Where were the trees? We've now seen that lists of tuples, permutations, combinations, and partitions were compiled rather early in human history, by interested and interesting researchers. Thus we've accounted for the evolution of the topics studied in Sections 7.2.1.1 through 7.2.1.5, and our story will be complete if we can trace the origins of tree generation, Section 7.2.1.6.

But the historical record of that topic before the advent of computers is virtually a blank page, with the exception of a few 19th-century papers by Arthur Cayley. Cayley's major work on trees, originally published in 1875 and reprinted on pages 427–460 of his *Collected Mathematical Papers*, Volume 4, was climaxed by a large foldout illustration that exhibited all the free trees with 9 or fewer unlabeled vertices. Earlier in that paper he had also illustrated the nine *oriented* trees with 5 vertices. The methods he used to produce those lists were quite complicated, completely different from Algorithm 7.2.1.6O and exercise 7.2.1.6–90. All free trees with up to 10 vertices were listed many years later by F. Harary and G. Prins [*Acta Math.* **101** (1958), 158–162], who also went up to $n = 12$ in the cases of free trees with no nodes of degree 2 or with no symmetries.

The trees most dearly beloved by computer scientists — binary trees or the equivalent ordered forests or nested parentheses — are however strangely absent from the literature. We saw in Section 2.3.4.5 that many mathematicians of the 1700s and 1800s had learned how to count binary trees, and we also know that the Catalan numbers C_n enumerate dozens of different kinds of combinatorial objects. Yet nobody seems to have published an actual *list* of the $C_4 = 14$ objects of order 4 in *any* of these guises, much less the $C_5 = 42$ objects of order 5, before 1950. (Except indirectly: The 42 genji-ko diagrams in (25) that have no intersecting lines turn out to be equivalent to the 5-node binary trees and forests. But this fact was not learned until the 20th century.)

There are a few isolated instances where authors of yore did prepare lists of $C_3 = 5$ Catalan-related objects. Cayley, again, was first; he illustrated the binary trees with 3 internal nodes and 4 leaves as follows in *Philosophical Magazine* **18** (1859), 374–378:

 (33)

(That same paper also illustrated another species of tree, equivalent to so-called weak orderings.) Then, in 1901, E. Netto listed the five ways to insert parentheses into the expression '$a + b + c + d$':

$$(a+b)+(c+d),\ \ [(a+b)+c]+d,\ \ [a+(b+c)]+d,\ \ a+[(b+c)+d],\ \ a+[b+(c+d)]. \qquad (34)$$

[*Lehrbuch der Combinatorik*, §122.] And the five permutations of $\{+1, +1, +1, -1, -1, -1\}$ whose partial sums are nonnegative were listed in the following way by Paul Erdős and Irving Kaplansky [*Scripta Math.* **12** (1946), 73–75]:

$$\begin{aligned}&1{+}1{+}1{-}1{-}1{-}1,\quad 1{+}1{-}1{+}1{-}1{-}1,\quad 1{+}1{-}1{-}1{+}1{-}1,\\ &\qquad\qquad 1{-}1{+}1{+}1{-}1{-}1,\quad 1{-}1{+}1{-}1{+}1{-}1.\end{aligned} \qquad (35)$$

Even though only five objects are involved, we can see that the orderings in (33) and (34) were basically catch-as-catch-can; only (35), which matches Algorithm 7.2.1.6P, was systematic and lexicographic.

We should also note briefly the work of Walther von Dyck, since many recent papers use the term "Dyck words" to refer to strings of nested parentheses. Dyck was an educator known for co-founding the Deutsches Museum in Munich, among other things. He wrote two pioneering papers about the theory of free groups [*Math. Annalen* **20** (1882), 1–44; **22** (1883), 70–108]. Yet the so-called Dyck words have at best a tenuous connection to his actual research: He studied the words on $\{x_1, x_1^{-1}, \ldots, x_k, x_k^{-1}\}$ that reduce to the empty string after repeatedly erasing adjacent letter-pairs of the forms $x_i x_i^{-1}$ or $x_i^{-1} x_i$; the connection with parentheses and trees arises only when we limit erasures to the first case, $x_i x_i^{-1}$.

Thus we may conclude that, although an explosion of interest in binary trees and their cousins occurred after 1950, such trees represent the only aspect of our story whose historical roots are rather shallow.

After 1950. Of course the arrival of electronic computers changed everything. The first computer-oriented publication about combinatorial generation methods was a note by C. B. Tompkins, "Machine attacks on problems whose variables are permutations" [*Proc. Symp. Applied Math.* **6** (1956), 202–205]. Thousands more were destined to follow.

Several articles by D. H. Lehmer, especially his "Teaching combinatorial tricks to a computer" in *Proc. Symp. Applied Math.* **10** (1960), 179–193, proved to be extremely influential in the early days. [See also *Proc. 1957 Canadian Math. Congress* (1959), 160–173; *Proc. IBM Scientific Computing Symposium on Combinatorial Problems* (1964), 23–30; and Chapter 1 of *Applied Combinatorial Mathematics*, edited by E. F. Beckenbach (Wiley, 1964), 5–31.] Lehmer represented an important link to previous generations. For example, Stanford's library records show that he had checked out Netto's *Lehrbuch der Combinatorik* in January of 1932.

The main publications relevant to particular algorithms that we've studied have already been cited in previous sections, so there is no need to repeat them here. But textbooks and monographs that first put pieces of the subject together in a coherent framework were also of great importance. Three books, in particular, were especially noteworthy with respect to establishing general principles:

- *Elements of Combinatorial Computing* by Mark B. Wells (Pergamon Press, 1971), especially Chapter 5.
- *Combinatorial Algorithms* by Albert Nijenhuis and Herbert S. Wilf (Academic Press, 1975). A second edition was published in 1978, containing additional material, and Wilf subsequently wrote *Combinatorial Algorithms: An Update* (Philadelphia: SIAM, 1989).
- *Combinatorial Algorithms: Theory and Practice* by Edward M. Reingold, Jurg Nievergelt, and Narsingh Deo (Prentice–Hall, 1977), especially the material in Chapter 5.

Robert Sedgewick compiled the first extensive survey of permutation generation methods in *Computing Surveys* **9** (1977), 137–164, 314. Carla Savage's survey article about Gray codes in *SIAM Review* **39** (1997), 605–629, was another milestone.

We noted above that algorithms to generate Catalan-counted objects were not invented until computer programmers developed an appetite for them. The first such algorithms to be published were not cited in Section 7.2.1.6 because they have been superseded by better techniques; but it is appropriate to list them here. First, H. I. Scoins gave two recursive algorithms for ordered tree generation, in the same paper we have cited with respect to the generation of *oriented* trees [*Machine Intelligence* **3** (1968), 43–60]. His algorithms dealt with binary trees represented as bit strings that were essentially equivalent to Polish prefix notation or to nested parentheses. Then Mark Wells, in Section 5.5.4 of his book cited above, generated binary trees by representing them as noncrossing set partitions. And Gary Knott [*CACM* **20** (1977), 113–115] gave recursive ranking and unranking algorithms for binary trees, representing them via the inorder-to-preorder permutations $q_1 \ldots q_n$ of Table 7.2.1.6–3.

A recent introduction to the entire subject can be found in Chapters 2 and 3 of *Combinatorial Algorithms: Generation, Enumeration, and Search* by Donald L. Kreher and Douglas R. Stinson (CRC Press, 1999).

Frank Ruskey is preparing a book entitled *Combinatorial Generation* that will contain a thorough treatment and a comprehensive bibliography. He has made working drafts of several chapters available on the Internet.

EXERCISES

Many of the exercises below ask a modern reader to find and/or to correct errors in the literature of bygone days. The point is not to gloat over how smart we are in the 21st century; the point is rather to understand that even the pioneers of a subject can stumble. One good way to learn that a set of ideas is not really as simple as it might seem to today's computer scientists and mathematicians is to observe that some of the world's leading thinkers had to struggle with the concepts when they were new.

1. [*15*] Does the subject of computing arise in the *I Ching*?

▸ **2.** [*M30*] (*The genetic code.*) DNA molecules are strings of “nucleotides” on the 4-letter alphabet $\{\mathtt{T},\mathtt{C},\mathtt{A},\mathtt{G}\}$, and most protein molecules are strings of “amino acids” on the 20-letter alphabet $\{\mathrm{A},\mathrm{C},\mathrm{D},\mathrm{E},\mathrm{F},\mathrm{G},\mathrm{H},\mathrm{I},\mathrm{K},\mathrm{L},\mathrm{M},\mathrm{N},\mathrm{P},\mathrm{Q},\mathrm{R},\mathrm{S},\mathrm{T},\mathrm{V},\mathrm{W},\mathrm{Y}\}$. Three consecutive nucleotides xyz form a “codon,” and a strand $x_1y_1z_1x_2y_2z_2\ldots$ of DNA specifies the protein $f(x_1,y_1,z_1)f(x_2,y_2,z_2)\ldots$, where $f(x,y,z)$ is the element in row z and column y of matrix x in the array

$$\begin{pmatrix} \mathrm{F} & \mathrm{S} & \mathrm{Y} & \mathrm{C} \\ \mathrm{F} & \mathrm{S} & \mathrm{Y} & \mathrm{C} \\ \mathrm{L} & \mathrm{S} & - & - \\ \mathrm{L} & \mathrm{S} & - & \mathrm{W} \end{pmatrix} \quad \begin{pmatrix} \mathrm{L} & \mathrm{P} & \mathrm{H} & \mathrm{R} \\ \mathrm{L} & \mathrm{P} & \mathrm{H} & \mathrm{R} \\ \mathrm{L} & \mathrm{P} & \mathrm{Q} & \mathrm{R} \\ \mathrm{L} & \mathrm{P} & \mathrm{Q} & \mathrm{R} \end{pmatrix} \quad \begin{pmatrix} \mathrm{I} & \mathrm{T} & \mathrm{N} & \mathrm{S} \\ \mathrm{I} & \mathrm{T} & \mathrm{N} & \mathrm{S} \\ \mathrm{I} & \mathrm{T} & \mathrm{K} & \mathrm{R} \\ \mathrm{M} & \mathrm{T} & \mathrm{K} & \mathrm{R} \end{pmatrix} \quad \begin{pmatrix} \mathrm{V} & \mathrm{A} & \mathrm{D} & \mathrm{G} \\ \mathrm{V} & \mathrm{A} & \mathrm{D} & \mathrm{G} \\ \mathrm{V} & \mathrm{A} & \mathrm{E} & \mathrm{G} \\ \mathrm{V} & \mathrm{A} & \mathrm{E} & \mathrm{G} \end{pmatrix}.$$

(Here $(\mathtt{T},\mathtt{C},\mathtt{A},\mathtt{G}) = (1,2,3,4)$; for example, $f(\mathtt{CAT})$ is the element in row 1 and column 3 of matrix 2, namely H.) Encoding proceeds until a codon leads to the stopper ‘$-$’.

a) Show that there is a simple way to map each codon into a hexagram of the *I Ching*, with the property that the 21 possible outcomes $\{\mathrm{A},\mathrm{C},\mathrm{D},\ldots,\mathrm{W},\mathrm{Y},-\}$ correspond to 21 *consecutive* hexagrams of the King Wen ordering (1).

b) Is that a sensational discovery?

3. [*20*] What is the millionth meter that has 30 beats, in colex ordering analogous to (2)? What is the rank of ⌣⌣⌣—⌣——⌣⌣⌣⌣——⌣⌣⌣⌣⌣⌣⌣—⌣—?

4. [*19*] Analyze the imperfections of Donnolo's list of permutations in Table 1.

5. [*16*] What's wrong with Kircher's list of five-note permutations in (7)?

6. [*25*] Mersenne published a table of the first 64 factorials on pages 108–110 of his *Traitez de la Voix et des Chants* (1636). His value for 64! was $\approx 2.2\times 10^{89}$; but it should have been $\approx 1.3\times 10^{89}$. Find a copy of his book and try to figure out where he erred.

7. [*20*] What permutations of $\{1,2,3,4,5\}$ are "alive" and "dead" according to Seki's rules (8) and (9)?

▸ **8.** [*M27*] Make a patch to (9) so that Seki's procedure will be correct.

9. [*15*] From (11), deduce the Arabic way to write the Arabic numerals $(0,1,\ldots,9)$.

▸ **10.** [*HM27*] In Ludus Clericalis, what is the expected number of times the three dice are rolled before all possible virtues are acquired?

11. [*21*] Decipher Llull's vertical table at the right of Fig. 45. What 20 combinatorial objects does it represent? *Hint:* Don't be misled by typographic errors.

12. [*M20*] Relate Schillinger's universal cycle (13) to the universal cycle of Poinsot in exercise 7.2.1.3–106.

13. [*21*] What should van Schooten have written, instead of (17)? Give also the corresponding tableau for combinations of the multiset $\{a,a,a,b,b,c\}$.

▸ **14.** [*20*] Complete the following sequence, from §95 of Izquierdo's *De Combinatione*:

ABC ABD ABE ACD ACE ACB ADE ADB ADC AEB . . . .

15. [*20*] (Nārāyaṇa Paṇḍita, 1356.) Design an algorithm to generate all compositions of n into parts $\le q$, namely all ordered partitions $n = a_1 + \cdots + a_t$, where $1 \le a_j \le q$ for $1 \le j \le t$ and t is arbitrary. Illustrate your method when $n = 7$ and $q = 3$.

16. [*HM27*] Analyze the algorithm of exercise 15.

17. [*10*] Trick question: Leibniz published his *Dissertatio de Arte Combinatoria* in 1666. Why was that a particularly auspicious year, permutationwise?

18. [*17*] In which of Puteanus's verses (20) is 'tibi' treated as ⌣— instead of ⌣⌣?

19. [*M25*] To commemorate the visit of three illustrious noblemen to Dresden in 1617, a poet published 1617 permutations of the hexameter verse

Dant tria jam Dresdæ, ceu sol dat, lumina lucem.

"Three give now to Dresden, as the sun gives, lights to light." [Gregor Kleppis, *Proteus Poeticus* (Leipzig: 1617).] How many permutations of those words would actually scan properly? *Hint:* The verse has dactyls in the first and fifth feet, spondees elsewhere.

20. [*HM30*] Let $f(p,q,r;s,t)$ be the number of ways to make (o^p, o^q, o^r) by concatenating the strings $\{s\cdot o, t\cdot oo\}$, when $p+q+r = s+2t$. For example, $f(2,3,2;3,2) = 5$ because the five ways are

$$(o|o, o|oo, oo),\quad (o|o, oo|o, oo),\quad (oo, o|o|o, oo),\quad (oo, o|oo, o|o),\quad (oo, oo|o, o|o).$$

a) Show that $f(p,q,r;s,t) = [u^p v^q w^r z^s]\, 1/((1-zu-u^2)(1-zv-v^2)(1-zw-w^2))$.
b) Use the function f to enumerate the scannable permutations of (19), subject to the additional condition that the fifth foot doesn't begin in the middle of a word.
c) Now enumerate the remaining cases.

▸ **21.** [*M40*] Look up the original discussions of the tot-tibi problem that were published by Prestet, Wallis, Whitworth, and Hartley. What errors did they make?

22. [*20*] What order of the 52 genji-ko diagrams corresponds to Algorithm 7.2.1.5H?

▸ **23.** [*23*] Early in the 1800s, Toshiaki Honda gave a recursive rule for generating all partitions of $\{1,\ldots,n\}$. His algorithm produced them in the following order when $n=4$:

Can you guess the corresponding order for $n=5$? *Hint:* See (26).

24. [*15*] The 16th-century author of *The Arte of English Poesie* was interested only in rhyme schemes that are "complete" in the sense of exercise 7.2.1.5–35; in other words, every line should rhyme with at least one other. Furthermore, the scheme should be "indecomposable" in the sense of exercise 7.2.1.2–100: A partition like $12|345$ decomposes into a 2-line poem followed by a 3-line poem. And the scheme shouldn't consist trivially of lines that all rhyme with each other. Under these conditions, is (28) a complete list of 5-line rhyme schemes?

▸ **25.** [*HM25*] How many n-line rhyme schemes satisfy the constraints of exercise 24?

▸ **26.** [*HM31*] The set partition $14|25|36$ can be represented by a genji-ko diagram such as ; but every such diagram for this partition must have at least three places where lines cross, and crossings are sometimes considered undesirable. How many partitions of $\{1,\ldots,n\}$ have a genji-ko diagram in which the lines cross at most once?

▸ **27.** [*25*] Let a, b, and c be prime numbers. John Wallis listed all possible factorizations of a^3b^2c as follows: *cbbaaa*, *cbbaa* · *a*, *bcaaa* · *b*, *bbaaa* · *c*, *cbba* · *aa*, *cbba* · *a* · *a*, *cbaa* · *ba*, *cbaa* · *b* · *a*, *bbaa* · *ca*, *bbaa* · *c* · *a*, *caaa* · *bb*, *caaa* · *b* · *b*, *baaa* · *cb*, *baaa* · *c* · *b*, *cbb* · *aaa*, *cbb* · *aa* · *a*, *cbb* · *a* · *a* · *a*, *cba* · *baa*, *cba* · *ba* · *a*, *cba* · *aa* · *b*, *cba* · *b* · *a* · *a*, *bba* · *caa*, *bba* · *ca* · *a*, *bba* · *aa* · *c*, *bba* · *c* · *a* · *a*, *caa* · *bb* · *a*, *caa* · *ba* · *b*, *caa* · *b* · *b* · *a*, *baa* · *cb* · *a*, *baa* · *ca* · *b*, *baa* · *ba* · *c*, *baa* · *c* · *b* · *a*, *aaa* · *cb* · *b*, *aaa* · *bb* · *c*, *aaa* · *c* · *b* · *b*, *cb* · *ba* · *aa*, *cb* · *ba* · *a* · *a*, *cb* · *aa* · *b* · *a*, *cb* · *b* · *a* · *a* · *a*, *bb* · *ca* · *aa*, *bb* · *ca* · *a* · *a*, *bb* · *aa* · *c* · *a*, *bb* · *c* · *a* · *a* · *a*, *ca* · *ba* · *ba*, *ca* · *ba* · *b* · *a*, *ca* · *aa* · *b* · *b*, *ca* · *b* · *b* · *a* · *a*, *ba* · *ba* · *c* · *a*, *ba* · *aa* · *c* · *b*, *ba* · *c* · *b* · *a* · *a*, *aa* · *c* · *b* · *b* · *a*, *c* · *b* · *b* · *a* · *a* · *a*. What algorithm did he use to generate them in this order?

▸ **28.** [*24*] In what order would Wallis have generated all factorizations of the number $abcde = 5\cdot 7\cdot 11\cdot 13\cdot 17$? Give your answer as a sequence of genji-ko diagrams.

29. [*M20*] What is the coefficient of $a_1^{i_1}a_2^{i_2}\ldots z^{m+n}$ in $(a_0z+a_1z^2+a_2z^3+\cdots)^m$? (See (29).)

30. [*20*] Compare de Moivre's and de Montmort's orders for partitions, (30) and (31), with Algorithm 7.2.1.4P.

31. [*21*] (R. J. Boscovich, 1748.) List all partitions of 20 for which all parts are 1, 7, or 10. Also design an algorithm that lists all such partitions of any given integer $n>0$.

SECTION 7.2.1.7

1. Perhaps under hexagram 21, “crunching” (䷔); however, the ancient commentators related this hexagram more to law enforcement than to the interaction of electrons.

2. (a) For the first nucleotide in the codon, let (`T`, `C`, `A`, `G`) be respectively represented by (, , ,); represent the second nucleotide, similarly, by (, , ,); represent the third by (, , ,); and superimpose those three representations. Thus, for example, hexagram number 34 is ䷡ = + + ; it represents the codon `TTC`, which maps to the amino acid F. Under this correspondence, hexagrams 34 through 54 inclusive map into the respective values (F, G, L, Q, W, D, S, –, P, Y, K, A, I, T, N, H, M, R, V, E, C). Moreover, the three hexagrams that map to ‘–’ are numbers 1, 9, and 41, namely ䷀, ䷈, and ䷨, which mean “creation”, “taming,” and “removal of excess” in the *I Ching*—all quite appropriate for the notion of completing a protein.

(b) Consider the $\binom{64}{6,6,6,4,4,4,4,4,3,3,2,2,2,2,2,2,2,2,2,1,1} \approx 2.3 \times 10^{69}$ ways to permute the elements of the $4 \times 4 \times 4$ genetic code array. Exactly

$$240288040217578979000399368196455132845166871875018555392000000 \approx 2.4 \times 10^{63}$$

of them contain at least one run of 21 distinct consecutive elements. [Using the principle of inclusion and exclusion one can show that any multiset $\{(n_1{+}1)\cdot x_1, \ldots, (n_r{+}1)\cdot x_r\}$ with r distinct elements and $n_r = 0$ has exactly

$$(n+1)\binom{n}{n_1,\ldots,n_r} r! - \sum_{k=1}^{r} (n+1-k)!\,k!\,(r-k)!\,a_k \sum_{\substack{0 \le d_1,\ldots,d_r \le 1\\ d_1+\cdots+d_r=k}} \binom{n-k}{n_1-d_1,\ldots,n_r-d_r}$$

such permutations, where $n = n_1 + \cdots + n_r$ and a_k is the number of indecomposable permutations with k elements (exercise 7.2.1.2–100).] Thus only about one out of every million permutations has the stated property.

But there are $4!^3\binom{6}{2,2,2} = 1244160$ ways to represent codons as in part (a), and most of them correspond to different permutations of the amino acids (except for interchanging the representations of `T` and `C` in third position).

Empirically, in fact, about 31% of all permutations of the 64 hexagrams turn out to have suitable codon mappings. Thus the construction in part (a) gives no reason to believe that the authors of the *I Ching* anticipated the genetic code in any way.

3. Since $F_{31} - 10^6 = F_{28} + F_{22} + F_{20} + F_{18} + F_{16} + F_{14} + F_9$, the millionth is

$$\smile\smile\smile\smile\smile\smile\smile-\smile\smile\smile-----\smile\smile\smile\smile\smile-\smile\smile.$$

Going the other way is easier: $F_{31} - (F_5 + F_8 + F_{10} + F_{16} + F_{18} + F_{27} + F_{30}) = 314159$.

4. One of the two appearances of במדיר on line 4 should be במידר; this glitch may simply be a typographical error. Similarly, one ירדמב on line 8 should be ירמדב. But the six cases with rightmost letters בד appear twice, in lines 3 and 4, while the cases with rightmost בר are missing. Donnolo himself must be responsible for this flaw.

5. The last one should have been , not .

6. The nth value m_n in Mersenne’s list agrees with $n!$ only for $1 \le n \le 13$ and $15 \le n \le 38$. Mersenne knew that $14! = 87178291200 \ne m_{14} = 8778291200$, because he inserted the missing ‘1’ in his personal copy of the book (now owned by the Bibliothèque Nationale; a facsimile was published in 1975). But the other errors in his table were not merely typographical, because they propagated into subsequent entries, except in the case of m_{50}: $m_{39} = 39! + 10^{26} - 10^{10}$; $m_{40} = 40m_{39}$; $m_{41} = 41m_{40} - 4 \cdot 10^{25} - 14 \cdot 10^{11}$; $m_n = nm_{n-1}$ for $n = 42$, 43, 44, 46, 47, 48, 49, 55, 60, and 62; $m_{50} = 50m_{49} + 10^{66}$; $m_{51} = 51 \cdot 50 \cdot m_{49}$. When he computed $m_{45} = 9 \cdot 45 \cdot m_{44} - 10^{40} + 10^{29}$, he apparently decided to take a shortcut, because it's easy to multiply by 5 or by 9; but he multiplied *twice* by 9. Most of his errors indicate an unreliable multiplication technique, which may have depended on an abacus: $m_{52} = 52m_{51} + 5 \cdot 10^{56} - 2 \cdot 10^{47} + 10^{34}$; $m_{53} = 53m_{52} - 4 \cdot 10^{29}$; $m_{54} = 54m_{53} + 10^{16}$; $m_{57} = 57m_{56} + 10^{33} + 10^{24}$; $m_{58} = 58m_{57} + 10^{67} - 10^{35} + 10^{32} + 11 \cdot 10^{26}$; $m_{59} = 59m_{58} + 10^{66} + 10^{49} - 10^{28}$; $m_{61} = 61m_{60} - 5 \cdot 10^{81}$; $m_{63} = 63m_{62} + 10^{82} - 10^{74}$; $m_{64} = 64m_{63} + 3 \cdot 10^{81} + 10^{67} + 2 \cdot 10^{38} - 2 \cdot 10^{33} - 10^{23}$.

The remaining case, $m_{56} \approx 10.912m_{55}$ is baffling; it is $\equiv 56m_{55}$ (modulo 10^{17}), but its other digits seem to satisfy neither rhyme nor reason. Can they be easily explained?

Notes: Athanasius Kircher must have copied from Mersenne when he tabulated $n!$ for $1 \le n \le 50$ on page 157 of his *Ars Magna Sciendi* (1669), because he repeated all of Mersenne's mistakes. Kircher did, however, list the values $10m_{14}$, $m_{45}/10$, and $10m_{49}$ instead of m_{14}, m_{45}, and m_{49}; perhaps he was trying to make the sequence grow more steadily. It is not clear who first calculated the correct value of 39!; exercise 1.2.5–4 tells the story of 1000!.

7. The basic permutations are 12345, 13254, 14523, 15432, 12453, 14235, 15324, 13542, 12534, 15243, 13425, 14352. But then we find that all 60 of the even permutations are both alive and dead, because (9) differs by an even permutation from (8). (Moreover, if we somehow repair the case $n = 5$, half of the live permutations for $n = 6$ will turn out to be odd.)

8. For example, we can replace (9) by

$$a_n a_3 \ldots a_{n-1} a_2 a_1,\ a_1 a_{n-1} \ldots a_n a_3 a_2,\ \ldots,\ a_{n-1} a_2 \ldots a_{n-2} a_1 a_n,$$

thus flipping the ends and cyclically shifting the other elements in the permutations of (8). This modification works because all permutations have the correct parity, and because the live and dead ones both have a_1 in every possible position. (We essentially have a dual Sims table for the alternating group, as in Eq. 7.2.1.2–(32); but our elements are named $(n, n-1, \ldots, 1)$ instead of $(0, 1, \ldots, n-1)$.)

A simpler way to generate permutations with the proper signs was published by É. Bézout [*Mémoires Acad. Royale des Sciences* (Paris, 1764), 292]: Each permutation $a_1 \ldots a_{n-1}$ of $\{1, \ldots, n-1\}$ yields n others, $a_1 \ldots a_{n-1} a_n - a_1 \ldots a_{n-2} a_n a_{n-1} + \cdots$.

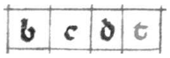
9. (٠, ١, ٢, ٣, ٤, ٥, ٦, ٧, ٨, ٩); or perhaps we should say (٩, ٨, ٧, ٦, ٥, ٤, ٣, ٢, ١, ٠). *Notes:* A different system was used for the index numbers of the equations; for example, ‘ز’ stood for 200. Moreover, it should be noted that (11) is actually a transcription of al-Samaw'al's work into *modern* Arabic; Ahmad and Rashed based their work on a 14th-century copy that used similar but older forms of the digits: (٥, ١, ٢, ٣, ۴, ۵, ٦, ٧, ٨, ٩). Al-Samaw'al himself may well have used numerals of an even earlier vintage.

10. If the 56 cases were equally likely, the answer would be $56H_{56} \approx 258.2$, as in the coupon collector's problem (exercise 3.3.2–8). But $(6, 30, 20)$ cases occur with the respective probabilities $(1/216, 1/72, 1/36)$; so the correct answer turns out to be

$$\int_0^\infty \bigl(1 - (1 - e^{-t/216})^6 (1 - e^{-t/72})^{30} (1 - e^{-t/36})^{20}\bigr)\, dt \approx 546.6,$$

about 42% of the upper bound $216H_{216}$. [See P. Flajolet, D. Gardy, and L. Thimonier, *Discrete Applied Math.* **39** (1992), 207–229.]

11. It tabulates the $\binom{6}{3} = 20$ combinations of $(\mathrm{b,c,d,B,C,D})$ taken three at a time; furthermore, they appear in lexicographic order if we regard $\mathrm{b < c < d < B < C < D}$. The letter t (t) means "shift from lowercase to uppercase." [See A. Bonner, *Selected Works of Ramon Llull* (Princeton: 1985), 596–597.] There are two typos: 'd' should be 'b' at the beginning of line 6; 'c' should be 'd' at the end of line 18. Line 1 would have been more consistent with the others if Llull had presented it as

 ;

but in that line, of course, no case shift was needed.

12. Multiply Poinsot's cycle by 5 (mod 7).

13. It's best to have just n lines when there are n different letters:

$$\frac{a.\ aa.\ aaa}{b.\ ab.\ aab.\ aaab.\ bb.\ abb.\ aabb.\ aaabb}$$

Then, assigning the weights $(a,b) = (1,4)$ gives the numbers 1 through 11 as in (18). (The first line of (16) should also be omitted.) Similarly, for $\{a,a,a,b,b,c\}$ we would implicitly give c the weight 12 and add the additional line

$$c.\ ac.\ aac.\ aaac.\ bc.\ abc.\ aabc.\ aaabc.\ bbc.\ abbc.\ aabbc.\ aaabbc.$$

[J. Bernoulli *almost* did it right in *Ars Conjectandi*, Part 2, Chapter 6.]

14. ABC ABD ABE ACD ACE ACB ADE ADB ADC AEB AEC AED BCD BCE BCA BDE BDA BDC BEA BEC BED BAC BAD BAE CDE CDA CDB CEA CEB CED CAB CAD CAE CBD CBE CBA DEA DEB DEC DAB DAC DAE DBC DBE DBA DCE DCA DCB EAB EAC EAD EBC EBD EBA ECD ECA ECB EDA EDB EDC. It's a genlex ordering (see Algorithm 7.2.1.3R), proceeding cyclically through the letters not yet used.

15. **N1.** [Initialize.] Set $r \leftarrow n$, $t \leftarrow 0$, and $a_0 \leftarrow 0$.

N2. [Advance.] While $r \ge q$, set $t \leftarrow t+1$, $a_t \leftarrow q$, and $r \leftarrow r - q$. Then if $r > 0$, set $t \leftarrow t+1$ and $a_t \leftarrow r$.

N3. [Visit.] Visit the composition $a_1 \ldots a_t$.

N4. [Find j.] Set $j \leftarrow t, t-1, \ldots,$ until $a_j \ne 1$. Terminate the algorithm if $j = 0$.

N5. [Decrease a_j.] Set $a_j \leftarrow a_j - 1$, $r \leftarrow t - j + 1$, $t \leftarrow j$; return to N2. ▮

For example, the compositions for $n = 7$ and $q = 3$ are 331, 322, 3211, 313, 3121, 3112, 31111, 232, 2311, 223, 2221, 2212, 22111, 2131, 2122, 21211, 2113, 21121, 21112, 211111, 133, 1321, 1312, 13111, 1231, 1222, 12211, 1213, 12121, 12112, 121111, 1132, 11311, 1123, 11221, 11212, 112111, 11131, 11122, 111211, 11113, 111121, 111112, 1111111.

Nārāyaṇa's sutras 79 and 80 gave essentially this procedure, but with the strings reversed (133, 223, 1123, . . .), because he preferred decreasing colex order. Curiously, he called this a "famous method, told by scholars of old dramatic art," although no references to prior descriptions are currently known except in the case $q = 2$.

16. The number V_n of visits is $F^{(q)}_{n+q-1} = \Theta(\alpha_q^n)$; see exercise 5.4.2–7. The number X_n of times step N4 tests $a_j = 1$ satisfies $X_n = X_{n-1} + \cdots + X_{n-q} + 1$, and we find $X_n = V_0 + \cdots + V_n = (qV_n + (q-1)V_{n-1} + \cdots + V_{n-q+1} - 1)/(q-1) = \Theta(V_n)$. The number Y_n of times step N2 sets $a_t \leftarrow q$ satisfies the same recurrence, and we find $Y_n = X_{n-q}$. And the number of times step N2 finds $r = 0$ turns out to be V_{n-q}.

17. It was MDCLXVI in Roman numerals, where M > D > C > L > X > V > I.

18. Lines 329 and 1022. (Puteanus included 139 such verses among his list of 1022.)

19. With 'tria' preceding 'lumina', there are $5! \times 2! \times (11, 12, 12, 16)$ ways having a dactyl in the (1st, 2nd, 3rd, 4th) foot, respectively; with 'lumina' preceding 'tria' there are $5! \times 2! \times (16, 12, 12, 11)$. So the total is 24480. [Leibniz considered this problem near the end of his *Dissertatio de Arte Combinatoria*, and came up with the answer 45870; but his argument was riddled with errors.]

20. (a) The generating function $1/((1 - zu - yu^2)(1 - zv - yv^2)(1 - zw - yw^2))$ is clearly equal to $\sum_{p,q,r,s,t\ge 0} f(p,q,r;s,t)\,u^p v^q w^r z^s y^t$.

(b) If 'tibi' is $\smile\smile$ and 'Virgo' is $-\,-$, the number is $3!\,3!$ times $\sum_{k=0}^{3}\bigl(f(2k+1, 6-2k, 2; 3, 3) + f(2k, 6-2k, 2; 2, 3)\bigr)$, namely $36\bigl((7{+}7) + (9{+}5) + (10{+}5) + (14{+}7)\bigr) = 2304$. Otherwise 'tibi' is $\smile -$, 'Virgo' is $- \smile$, and the number is $2!\,3!$ times $\sum_{k=0}^{3}\bigl(f(2k, 5-2k, 2; 3, 2) + f(2k, 6-2k, 1; 3, 2)\bigr)$, namely $12\bigl((7{+}6) + (5{+}4) + (4{+}4) + (0{+}6)\bigr) = 432$.

(c) The fifth foot begins with the second syllable of 'cælo', 'dotes', or 'Virgo'. Hence the additional number is $3!\,3!\sum_{k=0}^{2} f(2k, 5-2k, 2; 3, 2) = 36(7 + 5 + 4) = 576$, and the grand total is $2304 + 432 + 576 = 3312$.

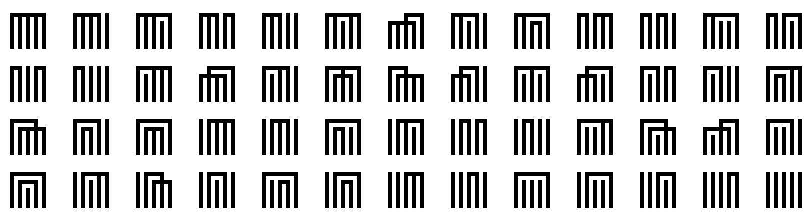
21. Let $\alpha \in \{\text{quot}, \text{sunt}, \text{tot}\}$, $\beta \in \{\text{cælo}, \text{dotes}, \text{Virgo}\}$, $\sigma = \text{sidera}$, and $\tau = \text{tibi}$. Prestet's analysis was essentially equivalent to that of Bernoulli, but he forgot to include the 36 cases $\alpha\alpha\alpha\tau\beta\beta\sigma\beta$. (In his favor one can say that those cases are poetically sterile; Puteanus found no use for them.) The 1675 edition of Prestet's book had also omitted all permutations that end with $\tau\beta$.

Wallis divided the possibilities into 23 types, $T_1 \cup T_2 \cup \cdots \cup T_{23}$. He claimed that his types 6 and 7 each yielded 324 verses; but actually $|T_6| = |T_7| = 252$, because his variable i should be 7, not 9. He also counted many solutions twice: $|T_3 \cap T_5| = 72$, $|T_2 \cap T_7| = |T_5 \cap T_7| = |T_3 \cap T_6| = |T_6 \cap T_{10}| = 36$, and $|T_{11} \cap T_{12}| = |T_{12} \cap T_{13}| = |T_{14} \cap T_{15}| = 12$. He missed the 36 possibilities $\alpha\beta\beta\alpha\sigma\alpha\tau\beta$ (19 of which were used by Puteanus). And he also missed all the permutations of exercise 20(c); Puteanus had used 250 of those 576. The Latin edition of Wallis's book, published in 1693, corrected several typographic errors in this section, but none of the mathematical mistakes.

Whitworth and Hartley omitted all cases with 'tibi' $= \smile -$ (see exercise 18), possibly because people's knowledge of classical hexameter was beginning to fade.

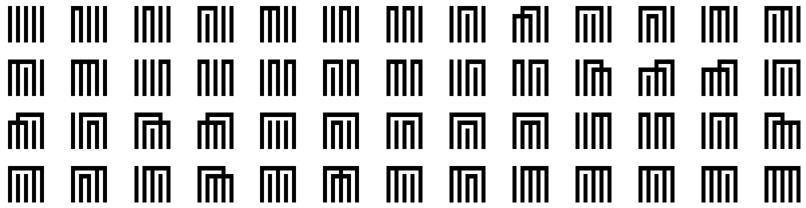
[Speaking of errors, Puteanus actually published only 1020 distinct permutations, not 1022, because lines 592 and 593 in his list were identical to lines 601 and 602. But he would have had no trouble finding two more cases — for example, by changing 'tot sunt' to 'sunt tot' in lines 252, 345, 511, 548, 659, 663, 678, 693, or 797.]

22. Reading each diagram left-to-right, so that $12|345 \leftrightarrow$, we get



23. His rule was: For $k = 0, 1, \ldots, n-1$, and for each combination $0 < j_1 < \cdots < j_k < n$ of $n-1$ things taken k at a time, visit all partitions of $\{1, \ldots, n-1\} \setminus \{j_1, \ldots, j_k\}$

together with the block $\{j_1, \ldots, j_k, n\}$. His order for $n = 5$ was:



But strictly speaking, the answer to this exercise is "No" — because Honda's rule is not complete until the order of the combinations is specified. He generated combinations in *colex* order (lexicographic on $j_t \ldots j_1$). Lexicographic order on $j_1 \ldots j_t$ would also be consistent with the list given for $n = 4$, but it would put  before . *Reference:* T. Hayashi, *Tôhoku Math. J.* **33** (1931), 332–337.

24. No; (28) misses $14|235$ (the top-bottom reflection of its second pattern).

25. Let a_n be the number of indecomposable partitions of $\{1, \ldots, n\}$, and let a'_n be the number that are both indecomposable and complete. These sequences begin $\langle a_1, a_2, \ldots\rangle = \langle 1, 1, 2, 6, 22, 92, 426, \ldots\rangle$, $\langle a'_1, a'_2, \ldots\rangle = \langle 0, 1, 1, 3, 9, 33, 135, \ldots\rangle$; and the answer to this exercise is $a'_n - 1$ for $n \geq 2$. It turns out that a_n is also the number of symmetric polynomials of degree n in noncommuting variables. [See M. C. Wolf, *Duke Math. J.* **2** (1936), 626–637, who also tabulated indecomposable partitions into k parts.]

If $A(z) = \sum_n a_n z^n$, and if $B(z) = \sum_n \varpi_n z^n$ is the non-exponential generating function for Bell numbers, we have $A(z)B(z) = B(z) - 1$, hence $A(z) = 1 - 1/B(z)$. And the result of exercise 7.2.1.5–35 implies that $\sum_n a'_n z^n = zA(z)/(1 + z - A(z)) = z(B(z)-1)/(1+zB(z))$. Unfortunately $B(z)$ has no especially nice closed form. Notice that indecomposable set partitions with $n > 1$ correspond to vacillating tableau loops with no three consecutive λs equal to zero (see exercise 7.2.1.5–27).

26. The problem is ambiguous because genji-ko diagrams are not well defined. Let's require all vertical lines of a block to have the same height; then, for example, $145|236$ has no single-crossing diagram because  is not allowed.

The number of partitions with no crossing is C_n (see exercise 7.2.1.6–26). For one crossing, the elements of the two blocks that cross must appear within the restricted growth sequence as either $x^i y x^j y^k$ or $x^i y^{j+1} x y^k$ or $x^i y^j x y^k x^l$, where $i, j, k, l > 0$.

Suppose the pattern is $x^i y x^j y^k$. The number of such partitions is

$$[z^{n-i-j-k-1}]\, C(z)^{i+j+k+2} = C_{(n-i-j-k-1)n}$$

by Eq. 7.2.1.6–(24). Summing on k gives $C_{(n-i-j-2)(n+1)}$; then summing on j and i gives $C_{(n-4)(n+3)}$.

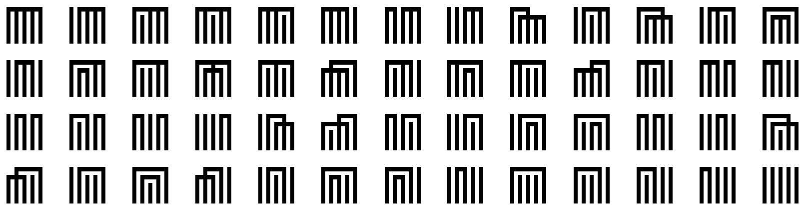
Similarly, the other two patterns contribute $C_{(n-5)(n+3)}$ and $C_{(n-5)(n+4)}$. The total number of single-crossing partitions is therefore $C_{(n-5)(n+3)} + C_{(n-4)(n+4)}$.

27. Order the divisors of *cbbaaa* by their number of prime factors and then colexicographically: $1 \prec a \prec b \prec c \prec aa \prec ba \prec ca \prec bb \prec cb \prec aaa \prec baa \prec caa \prec bba \prec cba \prec cbb \prec baaa \prec caaa \prec bbaa \prec cbaa \prec cbba \prec bbaaa \prec cbaaa \prec cbbaa \prec cbbaaa$. For every such divisor d, in decreasing order, let d be the first factor; recursively append all factorizations of *cbbaaa*$/d$ whose first factor is $\preceq d$.

If the divisors had been ordered lexicographically (namely $1 < a < aa < aaa < b < ba < \cdots < cbbaa < cbbaaa$), Wallis's algorithm would have been equivalent to

Algorithm 7.2.1.5M with $(n_1, n_2, n_3) = (1, 2, 3)$. He probably chose his more complicated ordering of the divisors because it tends to agree more closely with ordinary numerical order when $a \approx b \approx c$; for example, his ordering is precisely numerical when $(a, b, c) = (7, 11, 13)$. By generating the divisors according to his somewhat complex scheme, Wallis was essentially generating multiset combinations, which we noted in Section 7.2.1.3 are equivalent to bounded compositions. [*Reference: A Discourse of Combinations* (1685), 126–128, with two typographic errors corrected.]

28. The factorizations $edcba$, $edcb\cdot a$, $edca\cdot b$, $\ldots$, $e\cdot d\cdot c\cdot b\cdot a$ correspond respectively to



29. The coefficient is zero unless $i_1 + 2i_2 + \cdots = n$; in that case it is $\binom{m}{k} a_0^{m-k} \binom{k}{i_1, i_2, \ldots}$ where $k = i_1 + i_2 + \cdots$. (Consider $(a_0 z)^m$ times $(1 + (a_1/a_0)z + (a_2/a_0)z^2 + \cdots)^m$.)

30. The order produced by that algorithm is decreasing lexicographic, the reverse of (31), if we assume that partitions $a_1 \ldots a_k$ have $a_1 \geq \cdots \geq a_k$; de Moivre's was increasing *colexicographic*.

31. $20\cdot 1 = 7 + 13\cdot 1 = 2\cdot 7 + 6\cdot 1 = 10 + 10\cdot 1 = 10 + 7 + 3\cdot 1 = 2\cdot 10$. In general, Boscovich suggested starting with $n\cdot 1$ and computing the successor of $a\cdot 10 + b\cdot 7 + c\cdot 1$ as follows: If $c \geq 7$, the successor is $a\cdot 10 + (b+1)\cdot 7 + (c-7)\cdot 1$; otherwise if $c + 7b \geq 10$, the successor is $(a+1)\cdot 10 + (c + 7b - 10)\cdot 1$; otherwise stop.

INDEX AND GLOSSARY

When an index entry refers to a page containing a relevant exercise, see also the *answer* to that exercise for further information. An answer page is not indexed here unless it refers to a topic not included in the statement of the exercise.

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